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Two monopoles of one type and one of another

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Abstract: The metric on the moduli space of charge (2,1) SU(3) Bogomolny-Prasad-Sommerfield monopoles is calculated and investigated. The hyperkähler quotient construction is used to provide an alternative derivation of the metric. Various properties of the metric are derived using the hyperkähler quotient construction and the correspondence between BPS monopoles and rational maps. Several interesting limits of the metric are also considered.

Keywords: Solitons Monopoles and Instantons, Integrable Field Theories.

Contents

1. Introduction

This paper is about the metric on the moduli space of $(2,1)$-monopoles. This metric is calculated and examined. A $(2,1)$-monopole is a solution of the SU(3) Bogomolny equation

$$D_i\Phi = B_i\,, \tag{1.1}$$

where D_i is the adjoint representation $\mathfrak{su}(3)$ covariant derivative and B_i is a non-Abelian magnetic field which is the Hodge dual of the $\mathfrak{su}(3)$ field strength. The Higgs field Φ is a scalar field transforming under the adjoint representation of $\mathfrak{su}(3)$. The Higgs field at infinity is required to lie in the gauge orbit of

$$\Phi_\infty = i\begin{pmatrix} s_1 & & \\ & s_2 & \\ & & s_3 \end{pmatrix}, \tag{1.2}$$

where $s_1+s_2+s_3=0$ and, by convention, $s_1<s_2<s_3$. This condition on Φ gives a map from the large sphere at infinity into the quotient space

$$\mathrm{orbit}_{\mathrm{SU}(3)}\Phi_\infty = \mathrm{SU}(3)/\mathrm{U}(1)^2\,. \tag{1.3}$$

Since $\pi_2(\mathrm{SU}(3)/\mathrm{U}(1)^2)=\mathbb{Z}^2$, the moduli space of monopoles is divided into sectors labelled by two topological charges, k_1 and k_2, and a monopole with these charges is called a (k_1,k_2)-monopole. These charges appear in the asymptotic expansion of Φ. For large $r=|\mathbf{x}|$, Φ lies in the gauge orbit of

$$\Phi_\infty - \frac{i}{2r}\begin{pmatrix} k_1 & & \\ & k_2-k_1 & \\ & & -k_2 \end{pmatrix}. \tag{1.4}$$

$(k_1,0)$-monopoles are embeddings of k_1-monopoles; SU(2) monopoles with topological charge k_1. The embedding is essentially the trivial embedding of $\mathfrak{su}(2)$ into the upperleft 2×2 block of $\mathfrak{su}(3)$ [2]. The mass of a $(k_1,0)$-monopole is $4\pi M k_1$, where

$$M = s_2 - s_1\,. \tag{1.5}$$

In the same way, $(0,k_2)$-monopoles are k_2-monopoles with the $\mathfrak{su}(2)$ embedded into the bottomright 2×2 block of $\mathfrak{su}(3)$. The mass of a $(0,k_2)$-monopole is $4\pi m k_2$ where

$$m = s_3 - s_2\,. \tag{1.6}$$

In this way, there are two different types of SU(3) monopoles: one type corresponding to each U(1). A $(k_1,0)$-monopole or a $(0,k_2)$-monopole is made up of only one type of monopole and behaves like the corresponding SU(2) monopole. A (k_1,k_2)-monopole has mass $4\pi(k_1M+k_2m)$. It is not unreasonable to think of a $(2,1)$-monopole as being composed of two monopoles of one type and one of another.

BPS monopoles interact in different ways depending on whether they are of the same type or of different types. The metric on the moduli space of two SU(3) monopoles of different type, the $(1,1)$-monopole, is the Taub-NUT metric [8, 14, 31]. This metric can be derived from physical arguments since it is the kinetic Lagrangian for the electromagnetic and scalar interactions of point dyons. The metric for a $(2,0)$-monopole, that is an SU(3) monopole with two monopoles of the same type, is the Atiyah-Hitchin metric. This is the metric on the moduli space of a 2-monopole. When the monopoles are far apart, the metric is exponentially close to a Taub-NUT metric which can be interpreted as the kinetic Lagrangian for point dyons. When the monopoles are not far apart, their interactions can not be understood in this way. The metric on the moduli space of $(2,1)$-monopoles mixes these interaction types.

In this paper, the metric on the moduli space of $(2,1)$-monopoles is calculated from Nahm data. Nahm data are solutions to nonlinear matrix equations in a single variable $s \in (s_1, s_3]$. The moduli space of $(2,1)$ Nahm data is diffeomorphic to the moduli space of $(2,1)$-monopoles. The calculation done in this paper is of the metric on the moduli space of $(2,1)$ Nahm data. This is assumed to be equivalent to calculating the metric on the monopole moduli space. We will refer to this moduli space as $\mathcal{M}_{(2,1)}$.

The calculation is complicated and the resulting metric is not transparent. Nonetheless it is possible to examine geodesic submanifolds of the moduli space. By examining how the monopoles behave in these submanifolds, it is possible to infer properties of the monopole interactions. This is explained in section 5.2.

During the calculation of the metric itself, it is useful to consider some aspects of the corresponding physical picture. The metric on the moduli space is the kinetic Lagrangian for three interacting monopoles of two different types. It is possible to distinguish monopoles of different types and it is also possible to think of the two different monopole types separately. As explained below, each type corresponds to separate, though interdependent, parts of the Nahm data. There is a $(2,\ )$ part of the Nahm data with $s \in (s_1, s_2]$ and a $(\ ,1)$ part with $s \in [s_2, s_3]$. It is possible to think of these parts of the Nahm data as corresponding to different part of the $(2,1)$-monopole, a $(\ ,1)$-monopole and a $(2,\ )$-monopole.

The $(\ ,1)$ Nahm data are very simple; they consist of coordinates for the $(\ ,1)$-monopole. The conditions on the Nahm data include a matching condition at $s = s_2$. Because of this matching condition, the precise form of the $(2,\ )$ Nahm data depends on the $(\ ,1)$ Nahm data. This means that the precise shape of the $(2,\ )$-monopole depends on the position of the $(\ ,1)$-monopole. In fact, the further away the $(\ ,1)$-monopole is, the more the $(2,\ )$-monopole resembles a $(2,0)$-monopole.

What is remarkable is that up to group transformations, each $(2,\ )$-monopole configuration corresponds to an ellipsoid of $(\ ,1)$-monopole positions. In other words, the matching condition and gauge structure of the Nahm data is such that for a given class of gauge equivalent $(2,\ )$ Nahm data there is an ellipsoid of $(\ ,1)$ Nahm data.

However, though each $(2,\ )$-monopole on this ellipsoid is equivalent, the corresponding $(2,1)$-monopoles are all different, since the $(\ ,1)$-monopole is in a different position in each. This difference disappears in the limit where the $(\ ,1)$-monopole mass, m, is zero. If $m = 0$, the stabiliser of the asymptotic Higgs field (1.2) is $\mathrm{SU}(2)\times\mathrm{U}(1)$. This is the case of nonAbelian residual symmetry. This picture of nonAbelian residual symmetry as a massless limit of the Abelian residual symmetry is due to Lee, Weinberg and Yi [32].

In the massless case, there is only one topological charge; there is also a holomorphic charge which is preserved because of the complex structure. The holomorphic charge counts the number of massless monopoles. We will distinguish between holomorphic and topological charges by putting holomorphic charges in square brackets. In this notation, the $m = 0$ limit of a $(2,1)$-monopole is a $(2,[1])$-monopole. In this space of $(2,[1])$-monopoles, repositioning the notional $(\ ,1)$-monopole position is an isometry. In fact, this repositioning combined with the phase of the $(\ ,1)$-monopole constitutes an $\mathrm{SU}(2)$ isometric action on the $(2,[1])$-monopole moduli space.

The metric on this moduli space is calculated by Dancer in [9]. It is a twelve-dimensional moduli space. These twelve dimensions can be interpreted as four dimensions parameterising the overall centre of mass and position of the monopole, three dimensions for the $\mathrm{SU}(2)$ isometry, three for the $\mathrm{SO}(3)$ orientation and, finally, two dimensions usually called the separation and cloud parameters. These last two parameters describe the separation of the two massive monopoles and the distance from these to the ellipsoid of notional massless monopole positions. This ellipsoid is often called the cloud.

The $(2,1)$ metric can be calculated in two parts. The first part is due to the $(2,\ )$-monopole. This metric includes the effects of the $(\ ,1)$-monopole on the $(2,\ )$-monopole but does not include the $(\ ,1)$-monopole itself. The second part of the metric is due to the $(\ ,1)$-monopole. This is described in section 2. First, the Nahm data are introduced along with the groups acting on them. Gauge-invariant coordinates on $\mathcal{M}_{(2,1)}$ are then defined from the Nahm data and a set of one-forms defined from the exterior derivatives of these coordinates. Tangent vectors dual to these one-forms are calculated and the metric expressed in terms of the tangent vectors. The calculation of the explicit metric is presented in section 3. In section 4.1 another way of calculating the metric is discussed in which the physical description given above is very apparent. In this section, the hyperkähler quotient construction is used to construct the metric on $\mathcal{M}_{(2,1)}$ from the direct product of the $(2,[1])$-monopole and 1-monopole metrics.

Section 4 contains a detailed discussion of the hyperkähler quotient construction in the context of the $(2,1)$ metric. In section 5, we derive certain properties of the moduli space using rational maps. As mentioned above, section 5 also contains an investigation of the geodesic submanifolds of $\mathcal{M}_{(2,1)}$ which correspond to sets of monopole configurations with extra symmetry. Two asymptotic limits of the metric are calculated in section 6. In section 6.1, the asymptotic expression is calculated for large (, 1)-monopole separation and in section 6.2 the point dyon metric is discussed. Section 7 contains concluding remarks. We explain that we were motivated by two possible applications: Hanany-Witten theory and the calculation of the approximate metric for the SU(2) three-monopole with two monopoles close together and one far away. It is also noted that there is an obvious variation on previous calculations, the $([1], 2, [1])$-monopole metric for SU(4) monopoles. This metric is calculated in section 8.

1.1 Notation and conventions

The calculations involve the use of different indices running over different ranges. The following conventions are used. Latin indices i, j and k run from one to three. Greek indices run from zero to three or, in the case of λ, from one to four. Latin indices a, b and c are used for all other ranges, often one to eight or one to seven.

We use two conventions for the generators of SU(2). The Pauli matrices are written τ_i and satisfy $\tau_i\tau_j = \delta_{ij}\mathbf{1}_2 + i\epsilon_{ijk}\tau_k$, with $\mathbf{1}_2$ the 2×2 unit matrix and $\tau_3 = \mathrm{diag}(1, -1)$. For simplicity of notation, we also use the basis $e_i = -\frac{i}{2}\tau_i$ and the inner product $\langle\ ,\ \rangle = -2\mathrm{trace}\,(\ ,\ )$. This gives $\langle e_i, e_j\rangle = \delta_{ij}$ and $[e_i, e_j] = \epsilon_{ijk}e_k$.

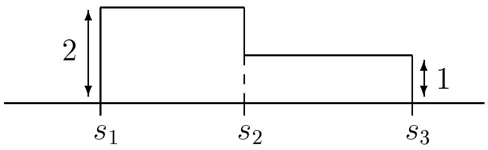
2. The moduli space of Nahm data

2.1 The Nahm data

Nahm data are quite complicated to describe. The Nahm data are matrix functions of a single variable over the finite interval defined by the eigenvalues of the asymptotic Higgs field. SU(3) Nahm data correspond to the asymptotic Higgs field Φ_∞ in (1.2) and so the interval is $(s_1, s_3]$. The interval is subdivided by the intermediate eigenvalues. In our case, the interval is subdivided by s_2 into $(s_1, s_2]$ and $[s_2, s_3]$. Each subinterval corresponds to a different monopole type and the dimension of the associated matrix functions is equal to the charge of that type of monopole. There are boundary conditions at the end points of the interval and matching conditions between matrices at boundaries between different subintervals. In each subinterval, the Nahm data satisfy the Nahm equations. From the Nahm data, the corresponding monopole fields may be constructed [36].

The $(2, 1)$ Nahm data are a quadruple $(\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$. Each $\mathcal{T}_\mu$ is a function of $s \in (s_1, s_3]$. For the lefthand interval, $s \in (s_1, s_2]$, $\mathcal{T}_\mu = T_\mu$ a 2×2 skew-hermitean matrix function satisfying the Nahm equation. For the righthand interval, $s \in [s_2, s_3]$, $\mathcal{T}_\mu = t_\mu$ is a 1×1 skew-hermitean matrix, in other words it is an imaginary number. In the construction of monopole fields from Nahm data, the (2,) fields are derived from the $s \in (s_1, s_2]$ Nahm data and the (, 1) fields from the $s \in [s_2, s_3]$ Nahm data. The reason that the lefthand Nahm data are 2×2 and the righthand Nahm data are

1×1 is that this is $(2,1)$ Nahm data. It can be represented by the diagram



(2.1)

The Nahm equations are

$$\frac{d\mathcal{T}_i}{ds} + [\mathcal{T}_0, \mathcal{T}_i] = [\mathcal{T}_j, \mathcal{T}_k]\,, \tag{2.2}$$

where $(i\ j\ k)$ is a cyclic permutation of $(1\ 2\ 3)$. For the $(2,1)$ Nahm data, this means that the lefthand Nahm data satisfy

$$\frac{dT_i}{ds} + [T_0, T_i] = [T_j, T_k]\,. \tag{2.3}$$

On the righthand Nahm data, the Nahm equations mean that the t_i are constants. The boundary conditions require T_i to have simple poles at $s = s_1$ whose matrix residues form an irreducible representation of $\mathfrak{su}(2)$. Finally, $\mathcal{T}_\mu$ is called continuous if

$$t_\mu(s_2) = (T_\mu(s_2))_{2,2}\,. \tag{2.4}$$

$\mathcal{T}_i$ are required to be continuous. $\mathcal{T}_0$ is not required to be continuous.

In this paper, we use the Nahm data to calculate the metric on the moduli space of monopoles. We do this by assuming that the moduli space of monopoles and the moduli space of Nahm data are isometric; in fact, this isometry has only been proven to exist for SU(2) [37]. The L^2 metric on the Nahm data is

$$\begin{aligned} ds^2 &= -\int_{s_1}^{s_3} \sum_\mu \mathrm{trace}\,(d\mathcal{T}_\mu d\mathcal{T}_\mu) ds \\ &= -\int_{s_1}^{s_2} \sum_\mu \mathrm{trace}\,(dT_\mu dT_\mu) ds - \int_{s_2}^{s_3} \sum_\mu dt_\mu dt_\mu ds\,, \end{aligned} \tag{2.5}$$

This is a hyperkähler metric with hyperkähler form

$$\omega = \int_{s_1}^{s_3} \mathrm{trace}\, d\mathcal{T} \wedge \overline{d\mathcal{T}} ds = \int_{s_1}^{s_2} \mathrm{trace}\, dT \wedge \overline{dT} ds + \int_{s_2}^{s_3} dt \wedge \overline{dt}\,, \tag{2.6}$$

where

$$\begin{aligned} d\mathcal{T} &= d\mathcal{T}_0 + Id\mathcal{T}_1 + Jd\mathcal{T}_2 + Kd\mathcal{T}_3\,, \\ \overline{d\mathcal{T}} &= d\mathcal{T}_0 - Id\mathcal{T}_1 - Jd\mathcal{T}_2 - Kd\mathcal{T}_3 \end{aligned} \tag{2.7}$$

and (I, J, K) are quaternions.

2.2 The Nahm construction

In order to construct monopole fields from Nahm data, the ADHMN equation must be solved. This is the ordinary differential equation

$$\left[\mathbf{1}_{2k}\frac{d}{ds} + \left(\mathbf{1}_k \otimes \sum_j x_j\sigma_j + i\sum_j \mathcal{T}_j \otimes \sigma_j\right)\right]\mathcal{V} = 0\,. \tag{2.8}$$

For $s \in (s_1, s_2]$, $k = 2$ and $\mathcal{V} = V(s; x_1, x_2, x_3)$, a complex 4-vector. (x_1, x_2, x_3) is the point in space at which the field is being constructed. For $s \in [s_2, s_3]$, $k = 1$ and $\mathcal{V} = v(s; x_1, x_2, x_3)$, a complex 2-vector. $\mathcal{V}$ is required to be continuous in the sense that the three and four components of $V(s_2; x_1, x_2, x_3)$ are required to be equal to $v(s_2; x_1, x_2, x_3)$. There is a three-dimensional space of these $\mathcal{V}$ and an orthonormal basis is chosen with respect to the inner product

$$(\mathcal{V}_1, \mathcal{V}_2) = \int_{s_1}^{s_2} V_1^\dagger V_2 ds + \int_{s_2}^{s_3} v_1^\dagger v_2 ds\,. \tag{2.9}$$

If $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ is such a basis the Higgs field is

$$(\Phi)_{ij} = (s\mathcal{V}_i, \mathcal{V}_j)\,, \tag{2.10}$$

with similar expressions for the gauge fields.

In unpublished work done in collaboration with Paul M. Sutcliffe, this construction has been performed for $(2,1)$-monopoles using the numerical scheme of [23]. It is observed that, generally, there appear to be three monopoles and that the $(\ ,1)$-monopole energy peak becomes lower and closer to the $(2, \ )$-monopole centre of mass as m is made smaller.

2.3 Group actions on the Nahm data

Another important aspect of the Nahm data is the group action. It is also difficult to describe succinctly. Generally,

$$\begin{aligned}\mathcal{T}_0 &\mapsto \mathcal{G}\mathcal{T}_0\mathcal{G}^{-1} - \frac{d\mathcal{G}}{ds}\mathcal{G}^{-1}\,,\\ \mathcal{T}_i &\mapsto \mathcal{G}\mathcal{T}_i\mathcal{G}^{-1}\end{aligned} \tag{2.11}$$

defines an action on the Nahm data. On the lefthand interval $\mathcal{G}$ is a U(2) matrix function G, on the righthand interval it is a U(1) function g. Let us define

$$G^{0,\star} = \left\{G \in C^\infty([s_1, s_2], \mathrm{U}(2)) : G(s_1) = \mathbf{1}_2\right\}, \tag{2.12}$$

and

$$g^{\star,0} = \left\{g \in C^\infty([s_2, s_3], \mathrm{U}(1)) : g(s_3) = 1\right\}. \tag{2.13}$$

$\mathcal{G}$ is called a gauge transformation if $\mathcal{G} \in \mathcal{G}^{0,0,0}$ where

$$\mathcal{G}^{0,0,0} = \left\{ \mathcal{G} = (G, g) : G \in G^{0,\star},\ g \in g^{\star,0},\ G(s_2) = \begin{pmatrix} 1 & 0 \\ 0 & g(s_2) \end{pmatrix} \right\}. \tag{2.14}$$

It should be noted that the gauge transformation satisfies a strong condition at $s = s_2$. As well as requiring continuity $(G(s_2))_{2,2} = g(s_2)$ the condition also fixes the other entries of $G(s_2)$.

This action is called a gauge action, because monopoles constructed from gauge equivalent Nahm data are gauge equivalent. The space of Nahm data is only diffeomorphic to the monopole moduli space, once the gauge action has been factored out. This factored space is called the moduli space of Nahm data.

The reason for writing the gauge group $\mathcal{G}^{0,0,0}$ with three superscripted zeros is that there are three boundary and junction conditions beyond the continuity condition $(G(s_2))_{2,2} = g(s_2)$. The first zero refers to the $G(s_1)$ being the identity, the second zero to the strong condition at $s = s_2$:

$$G(s_2) = \begin{pmatrix} 1 & 0 \\ 0 & g(s_2) \end{pmatrix}, \tag{2.15}$$

and the third zero to $g(s_3)$ being the identity. For similar groups in which the second or third condition is relaxed, the corresponding zero is replaced by a star. These groups are important, because group transformations which do not satisfy these two conditions nonetheless map Nahm data to Nahm data and are used frequently in the calculation of the metric. They are used to derive all the Nahm data from a specific ansatz solution and the group parameters are used as coordinates on the moduli space. Thus, the groups we will need are

$$\mathcal{G}^{0,\star,0} = \left\{ \mathcal{G} = (G, g) : G \in G^{0,\star},\ g \in g^{\star,0},\ (G(s_2))_{2,2} = g(s_2) \right\} \tag{2.16}$$

and

$$\mathcal{G}^{0,\star,\star} = \left\{ \mathcal{G} = (G, g) : G \in G^{0,\star},\ g \in g^{\star,\star},\ (G(s_2))_{2,2} = g(s_2) \right\}, \tag{2.17}$$

where

$$g^{\star,\star} = \{ g \in C^{\infty}([s_2, s_3], \mathrm{U}(1)) \}. \tag{2.18}$$

We will also use

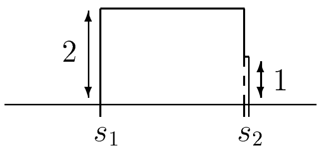
$$G^{0,0} = \left\{ G \in C^{\infty}([s_1, s_2], \mathrm{U}(2)) : G(s_1) = \mathbf{1}_2,\ G(s_2) = \mathbf{1}_2 \right\}. \tag{2.19}$$

The group $\mathcal{G}^{\star,0,0}$, in which the first condition is relaxed, is used later in this section when defining the rotational SO(3) action.

In the $(2,[1])$ case considered by Dancer, the asymptotic Higgs field has two equal eigenvalues, this means

$$\Phi_{\infty} = i \begin{pmatrix} s_1 & & \\ & s_2 & \\ & & s_2 \end{pmatrix}, \tag{2.20}$$

which has little group U(2). The $(2,[1])$ case has minimal symmetry breaking; SU(3) is broken to U(2). Since $m=0$, the Nahm data may be represented as



(2.21)

The gauge action on this Nahm data is given by $G^{0,0}$ and the unbroken U(2) is $G^{0,\star}/G^{0,0}$. In the $(2,[1])$ moduli space this U(2) is an isometry. There is a similar U(2) action when m is not zero. However, it is not an isometry. This U(2) action is $\mathcal{G}^{0,\star,\star}/\mathcal{G}^{0,0,0}$. The U(2) isometry found in the $(2,[1])$ case has split into two parts. There is an S^3 part at $s=s_2$ which may be written as the coset $\mathcal{G}^{0,\star,0}/\mathcal{G}^{0,0,0}$ and a U(1) which may be written as the coset $\mathcal{G}^{0,0,\star}/\mathcal{G}^{0,0,0}$.

It is useful to discuss the physical meaning of the group transformations. The $(\,,1)$-monopole has a well defined position given by

$$\mathbf{r} = -i(t_1,t_2,t_3)\,. \tag{2.22}$$

This is fixed under the maximal torus of the U(2)= $\mathcal{G}^{0,\star,\star}/\mathcal{G}^{0,0,0}$ action consisting of group elements which are diagonal at $s=s_2$. The action of this maximal torus is isometric. The orbit of $\mathbf{r}$ under the U(2) action is an ellipsoid; this will be clear when we write down solutions of the Nahm equations in section 2.5. This ellipsoid corresponds to the coset $\mathrm{U}(2)/\mathrm{U}(1)^2$. Thus the group action consists of an isometric maximal torus and a nonisometric coset which moves the $(\,,1)$-monopole around on an ellipsoid.

There is also an SO(3) action which rotates the whole monopole in space. The Nahm data are acted on by rotating $(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)$ as a three-vector. Precisely, if $A\in\mathcal{G}^{\star,0,0}$ and $A(s_1)$ maps to the SO(3) matrix (A_{ij}), this action is defined by

$$\begin{aligned}
T_0 &\to AT_0A^{-1} - \frac{dA}{ds}A^{-1}\,,\\
T_i &\to A\left(\sum_j A_{ij}T_j\right)A^{-1}\,,\\
t_0 &\to t_0 - \frac{1}{a}\frac{da}{ds},\\
t_i &\to \sum_j A_{ij}t_j\,,
\end{aligned} \tag{2.23}$$

where $(A(s_2))_{2,2} = a(s_2)$. Thus, the SO(3) action rotates the $(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)$ without disturbing the matrix residues at $s=s_1$.

Finally, there is a translational $\mathbb{R}^3$ action on the Nahm data. It is given on the Nahm data by

$$\begin{aligned} T_i &\mapsto T_i + i\lambda_i \mathbf{1}_2 \,, \\ t_i &\mapsto t_i + i\lambda_i \,. \end{aligned} \tag{2.24}$$

This action is an isometry and corresponds to a translation in space of the whole monopole.

2.4 The centre of mass

The overall motion of a monopole is free and the (2,1) moduli space decomposes as

$$\mathcal{M}_{(2,1)} = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}^0_{(2,1)}}{\mathbb{Z}} \,, \tag{2.25}$$

where the $\mathbb{R}^3$ and $\mathbb{R}$ correspond to the overall position and overall phase. $\mathcal{M}^0_{(2,1)}$ is called the centred moduli space and contains the nontrivial structure of $\mathcal{M}_{(2,1)}$. Calculating the metric on $\mathcal{M}^0_{(2,1)}$ is the main business of this paper and this calculation is simplified by the observation that the traces of the T_μ can be set to zero. This is explained later in this section.

It is easy to see from the Nahm equations (2.3) that the traces of the T_i are constant. This is something they have in common with the righthand data; t_i are also constant. A gauge can be chosen so that both $\mathrm{trace}\, T_0$ and t_0 are constants. This does not define a specific gauge or specific constants: neither constant is gauge invariant. The invariant combination is $M \mathrm{trace}\, T_0 + m t_0$ and this combination plays a significant role in the calculation done in this section.

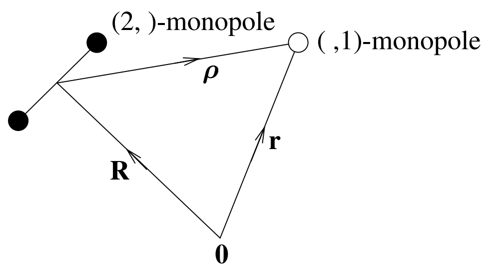


Figure 1: An illustration of the definition of the various separation vectors.

The metric can now be rewritten in terms of traceless data and the traces themselves: the Nahm data T_μ are replaced by $T_\mu + iR_\mu \mathbf{1}_2$ where T_μ is now traceless. Defining $\mathbf{r}$ as above, (2.22), and $r_0 = -it_0$, the metric (2.5) becomes

$$ds^2 = 2M \sum_\mu dR_\mu dR_\mu + m \sum_\mu dr_\mu dr_\mu - \int_{s_1}^{s_2} \sum_\mu \mathrm{trace}\, dT_\mu dT_\mu ds \,. \tag{2.26}$$

Furthermore, the action of the translation action and the fact that the (,1)-monopole position is $\mathbf{r}$, suggests that the centre of mass of the (2,)-monopole is positioned at $\mathbf{R}$. This is illustrated in figure 1, along with the relative position vector

$$\rho_i = r_i - R_i \,. \tag{2.27}$$

This implies that the centre of mass is given by

$$\frac{2MR_i + mr_i}{2M+m} = R_i + \frac{m}{2M+m}\rho_i \tag{2.28}$$

and, in fact,

$$ds^2 = ds^2_{\text{centre}} + \frac{2Mm}{2M+m}\sum_\mu d\rho_\mu d\rho_\mu - \int_{s_1}^{s_2}\sum_\mu \text{trace}\, dT_\mu dT_\mu ds\,, \tag{2.29}$$

where $\rho_0 = R_0 - r_0$ and

$$ds^2_{\text{centre}} = (2M+m)d\left(\frac{2MR_\mu + mr_\mu}{2M+m}\right) d\left(\frac{2MR_\mu + mr_\mu}{2M+m}\right). \tag{2.30}$$

Thus, the metric separates into two terms, one of which is flat. In section 3 the metric on the term which is not flat is calculated. The ρ_i satisfy the same continuity conditions with respect to the traceless Nahm data as r_i do with respect to the general Nahm data. This means that the metric we want is the metric on the space of traceless Nahm data.

Thus, in section 3 the metric is calculated on the space of Nahm data with traceless T_i and T_0 and with the mass m replaced by the reduced mass $2Mm/(2M+m)$ as in (2.29). For simplicity, rather than changing to the reduced mass, we will continue to use $m = s_3 - s_2$ and to denote the space by $\mathcal{M}^0_{(2,1)}$. In fact, it is demonstrated above that in the metric on $\mathcal{M}^0_{(2,1)}$ the mass m must be replaced by the reduced mass.

The fixing of the centre of mass is discussed from a different perspective in section 4, where it is fixed using a hyperkähler quotient.

2.5 Solving the Nahm equations

The Nahm equations on the lefthand interval are easy to solve. The ansatz

$$\begin{aligned} T_0(s) &= 0\,, \\ T_i(s) &= -\frac{i}{2}f_i(s)\tau_i = f_i(s)e_i\,, \end{aligned} \tag{2.31}$$

reduces them to the well-known Euler-Poinsot equations

$$\frac{d}{ds}f_i(s) = f_j(s)f_k(s)\,, \tag{2.32}$$

where $(i\ j\ k)$ is an cyclic permutation of $(1\ 2\ 3)$. Assuming $[f_1(s)]^2 \le [f_2(s)]^2 \le [f_3(s)]^2$ the solutions are the Euler top functions

$$\begin{aligned} f_1(s) &= -\frac{D\,\text{cn}_k D(s-s_1)}{\text{sn}_k D(s-s_1)}\,, \\ f_2(s) &= -\frac{D\,\text{dn}_k D(s-s_1)}{\text{sn}_k D(s-s_1)}\,, \\ f_3(s) &= -\frac{D}{\text{sn}_k D(s-s_1)}\,. \end{aligned} \tag{2.33}$$

These solutions have a pole of the correct form at $s = s_1$. k is the elliptic parameter and $0 \le k \le 1$. In order for the Nahm data to be nonsingular inside the interval $(s_1, s_2]$, we must have $D < 2K(k)/M$, where $K(k)$ is the usual complete elliptic integral of the first kind. Thus, the ansatz yields a two-parameter space of solutions. The Nahm data t_i on the righthand interval are determined by the continuity condition (2.4) at $s = s_2$.

Of course, not all Nahm data are produced by this ansatz. However, all the required Nahm data can be produced by acting on this two-parameter ansatz space with the group actions described in section 2.3 and below. In other words, D and k label group orbits and the complete orbit of the ansatz space is the whole manifold. The orbits are not identical, however, and the ansatz space is not a manifold.

There are two group actions on the Nahm data. First, there is the SO(3) rotation action given in (2.23). On uncentred Nahm data, we also have an action of $\mathcal{G}^{0,\star,\star}/\mathcal{G}^{0,0,0} = (\mathrm{U}(2) \times \mathrm{U}(1))/\mathrm{U}(1)$. One effect of centring the Nahm data by the procedure of section 2.4 is that the group action must have unit determinant. Thus, the correct group action on the centred Nahm data is $\mathrm{SO}(3) \times \mathrm{S}\left(\mathcal{G}^{0,\star,\star}/\mathcal{G}^{0,0,0}\right) = \mathrm{SO}(3) \times \mathrm{SU}(2)$. After a general transformation of this type, the Nahm data can be put in the form

$$\begin{aligned}(T_0, T_i) &= \left(-G\frac{dA}{ds}A^{-1}G^{-1} - \frac{dG}{ds}G^{-1}, GA(\sum_j A_{ij} f_j(s) e_j)A^{-1}G^{-1}\right),\\ (t_0, t_i) &= \left(-\frac{da}{ds}a^{-1} - \frac{dg}{ds}g^{-1}, (T_i(s_2))_{2,2}\right).\end{aligned} \tag{2.34}$$

This is an eight-dimensional space of Nahm data. This eight-dimensional space is assumed to be isometric to the relative moduli space of $(2,1)$-monopoles. We use $\mathcal{M}^0_{(2,1)}$ to denote both of these moduli spaces.

2.6 Coordinates and one-forms

Eight coordinates on $\mathcal{M}^0_{(2,1)}$ are now required. In order to define such coordinates, an explicit representation of the group action at $s = s_2$ in terms of Euler angles is needed. This is given by

$$G(s_2) = \begin{pmatrix} \cos\frac{\theta}{2} e^{-\frac{i}{2}(\chi+\phi)} & -\sin\frac{\theta}{2} e^{\frac{i}{2}(\chi-\phi)} \\ \sin\frac{\theta}{2} e^{-\frac{i}{2}(\chi-\phi)} & \cos\frac{\theta}{2} e^{\frac{i}{2}(\chi+\phi)} \end{pmatrix}. \tag{2.35}$$

Since the group action on the Nahm data is an adjoint action, it descends from SU(2) to SO(3) action under the usual homomorphism

$$G(s_2) \mapsto E_{ij} = \frac{1}{2}\mathrm{trace}\,\left(\tau_i G(s_2) \tau_j G(s_2)^\dagger\right). \tag{2.36}$$

In terms of the Euler angles, the matrix E is

$$\begin{pmatrix} \cos\theta\cos\chi\cos\phi - \sin\phi\sin\chi & -\cos\theta\cos\phi\sin\chi - \cos\chi\sin\phi & \cos\phi\sin\theta \\ \cos\phi\sin\chi + \cos\theta\cos\chi\sin\phi & -\cos\theta\sin\phi\sin\chi + \cos\chi\cos\phi & \sin\theta\sin\phi \\ -\cos\chi\sin\theta & \sin\theta\sin\chi & \cos\theta \end{pmatrix} . \tag{2.37}$$

The Nahm data are acted on by both this group action and the rotation action defined in (2.23). After acting with general elements of these groups, the traceless Nahm data are

$$\begin{aligned} T_1(s) &= \sum_{i,j} A_{1i} f_i(s) E_{ji} e_j \,, \\ T_2(s) &= \sum_{i,j} A_{2i} f_i(s) E_{ji} e_j \,, \\ T_3(s) &= \sum_{i,j} A_{3i} f_i(s) E_{ji} e_j \,. \end{aligned}$$

Five of the coordinates introduced by Dancer on the (2,[1]) moduli space can be immediately adopted as coordinates on $\mathcal{M}^0_{(2,1)}$.

$$\begin{aligned} \alpha_1 &= \langle T_1, T_1\rangle - \langle T_2, T_2\rangle \,, \\ \alpha_2 &= \langle T_1, T_1\rangle - \langle T_3, T_3\rangle \,, \\ \alpha_3 &= \langle T_1, T_2\rangle \,, \\ \alpha_4 &= \langle T_1, T_3\rangle \,, \\ \alpha_5 &= \langle T_2, T_3\rangle \,. \end{aligned}$$

These coordinates are invariant under the gauge and group actions and independent of s. α_3, α_4 and α_5 are coordinates for the rotational action. In the $(2,[1])$ moduli space combinations of α_1 and α_2 are the separation and cloud parameters. They play a similar role here, except that the cloud parameter in the $m = 0$ case becomes a genuine separation parameter when m is not zero. It then corresponds to the separation of the (, 1)-monopole from the centre of the (2,)-monopole. The last three coordinates used in [9] are

$$\begin{aligned} \cos\alpha_6 &= \frac{\langle T_3, e_3\rangle}{||T_3||} \,, \\ \cos\alpha_7 &= \frac{\langle [T_3, T_2], [e_3, T_3]\rangle}{||[T_3, T_2]||\,||[e_3, T_3]||} \,, \\ \sin\alpha_8' &= \frac{\langle T_3, e_2\rangle}{||[T_3, e_3]||} \,, \end{aligned} \tag{2.38}$$

where, in each case, the righthand side is evaluated at $s = s_2$.

Two of these coordinates, α_6 and α_7, are gauge-invariant, that is, invariant under a general element of $\mathcal{G}^{0,0,0}$. If $A = \mathbf{1}_3$ they correspond to the θ and χ Euler angles which determine the position of the (,1)-monopole on the ellipsoid. Thus they parameterise an S^2 surface acted on by SU(2).

The third coordinate, α_8', is not gauge invariant. The diagonal U(1) subgroup of the SU(2) action corresponds to the relative phase of the two monopoles. The overall phase of the (2,)-monopole is given by α_8'. The expression for the phase of a $(\ldots,\ ,1,\ ,\ldots)$-monopole in the moduli space of $(1,1,\ldots,1)$-monopoles is $i\int t_0 ds$ [35]. Consequently, we try the combination

$$\alpha_8 = \alpha_8' + i\int_{s_2}^{s_3} t_0 ds\,. \tag{2.39}$$

This is a well-defined coordinate: direct calculation shows that it is gauge-invariant. It has a clear physical interpretation; the two terms are the phases of the (2,)-monopole and the (,1)-monopole in that order. This shows that the relative phase of the (,1)-monopole and the (2,)-monopole is changed by a group action but not by a gauge action. Furthermore, the normalisation of α_8 is correct because it is identified modulo 2π under the group action $\mathcal{G}^{0,0,\star}/\mathcal{G}^{0,0,0}$.

It should be noted that the coordinates above reduce to the coordinates used by Dancer [9] in the $m \to 0$ limit. This is a smooth limit.

2.7 One-forms and tangent vectors

Now that we have a full set of coordinates for $\mathcal{M}^0_{(2,1)}$ we can compute the metric. To do this, a basis of one-forms is derived by taking the exterior derivatives of the coordinates: α_a, $a = 1,\ldots,8$. A basis of tangent vectors orthogonal to these one-forms is then found and their inner product calculated. This gives a local expression for the metric.

Thus, it is now important to define a basis of physically motivated and computationally convenient one-forms and express the metric in terms of these. One-forms corresponding to α_1 to α_5 are defined as $\rho_a = d\alpha_a$ for $a = 1,\ldots,5$. To deal with the SU(2) group action we start by considering the set of left-invariant one-forms. These are

$$\begin{aligned}
\rho_6 &= \cos\alpha_8' d\alpha_6 + \sin\alpha_8' \sin\alpha_6 d\alpha_7\,,\\
\rho_7 &= -\sin\alpha_8' d\alpha_6 + \cos\alpha_8' \sin\alpha_6 d\alpha_7\,,\\
\rho_8' &= d\alpha_8' + \cos\alpha_6 d\alpha_7\,.
\end{aligned} \tag{2.40}$$

To perform the explicit calculation of the metric, matters are simplified greatly if we use the U(1) isometry to work at infinitesimal α_8'. It may be assumed that for all explicit calculations, we are working in this limit in which, neglecting infinitesimal terms, these expressions become

$$\begin{aligned}
\rho_6 &= d\alpha_6\,,\\
\rho_7 &= \sin\alpha_6 d\alpha_7\,,\\
\rho_8' &= d\alpha_8' + \cos\alpha_6 d\alpha_7\,.
\end{aligned} \tag{2.41}$$

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The reason ρ_8' appears with a prime is that we will have cause to modify it when we come to calculate the metric. We reserve the symbols ρ_a for $a = 1, \ldots, 8$ for the one-forms used in the calculation.

ρ_8' is well-defined. Although $\alpha_8 = \alpha_8' + i\int_{s_2}^{s_3} t_0 ds$ rather than either α_8' or $i\int_{s_2}^{s_3} t_0 ds$ alone is the well-defined coordinate, $d\alpha_8' + \cos\alpha_6 d\alpha_7$ and $d(i\int_{s_2}^{s_3} t_0 ds)$ are both well-defined one-forms. They each give gauge-invariant results when contracted with tangent vectors. The tangent vectors are discussed in the remainder of this section and the gauge invariance of the contraction can be seen by direct calculation in the case of $d(i\int_{s_2}^{s_3} t_0 ds)$. In the case of $d\alpha_8' + \cos\alpha_6 d\alpha_7$ gauge invariance follows from the fact it is the one-form used in the calculation of the $(2,[1])$ metric and this metric is isometric under $G^{0,\star}$.

The next problem is that of finding the tangent vectors to a point in $\mathcal{M}_{(2,1)}$. In the Nahm description, these tangent vectors are quadruples $(\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3)$ where each $\mathcal{Y}_\mu$ is a 2×2 skew-hermitean matrix function, Y_μ, on the lefthand interval and an imaginary function, y_μ, on the righthand interval. In other words, $Y_\mu(s) \in \mathfrak{u}(2)$ and $y_\mu(s) \in \mathfrak{u}(1)$. In order for these vectors to be tangent to $\mathcal{M}_{(2,1)}$, $\mathcal{Y}_i$ must satisfy the linearisation of the conditions satisfied by $\mathcal{T}_i$. This means that they satisfy the linearised Nahm equations. On the lefthand interval, the linearised Nahm equations are

$$\begin{aligned}
\frac{dY_1}{ds} + [Y_0, T_1] + [T_0, Y_1] &= [T_2, Y_3] + [Y_2, T_3]\,, \\
\frac{dY_2}{ds} + [Y_0, T_2] + [T_0, Y_2] &= [T_3, Y_1] + [Y_3, T_1]\,, \\
\frac{dY_3}{ds} + [Y_0, T_3] + [T_0, Y_3] &= [T_1, Y_2] + [Y_1, T_2]\,.
\end{aligned} \tag{2.42}$$

On the righthand interval the linearised Nahm equations require y_i to be constant. To preserve the tracelessness of the lefthand Nahm data $\mathrm{trace}\, Y_i$ must be set to zero. This is consistent since (2.42) implies that $\mathrm{trace}\, Y_i$ are constant. Furthermore, the $\mathcal{Y}_i$ must be continuous. This means that we have $y_i = (Y_i(s_2))_{2,2}$. Since the points of $\mathcal{M}_{(2,1)}$ correspond to gauge-equivalent sets of Nahm data, the tangent vectors must also be orthogonal to the gauge transformations. The inner product on the tangent space is given by

$$(\mathcal{X}, \mathcal{Z}) = -\int_{s_1}^{s_2} \sum_\mu \mathrm{trace}\, X_\mu Z_\mu ds - \int_{s_2}^{s_3} \sum_\mu x_\mu z_\mu ds\,, \tag{2.43}$$

where $\mathcal{X}$ and $\mathcal{Z}$ are two tangent vectors composed of the quadruples $(\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$ and $(\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3)$. The metric is derived from this inner product and it is with respect to this inner product that the tangent vectors and the gauge transformations are required to be orthogonal. An infinitesimal gauge transformation has the form

$$\left(\frac{d\Psi}{ds} + [T_0, \Psi], [T_1, \Psi], [T_2, \Psi], [T_3, \Psi]\right), \tag{2.44}$$

on the lefthand interval. Ψ is a $\mathfrak{u}(2)$ function which is zero at $s = s_1$ and proportional to diag$(0, 1)$ at $s = s_2$. On the righthand interval, the infinitesimal transformation has the form

$$\left(\frac{d\psi}{ds}, 0, 0, 0\right), \tag{2.45}$$

where ψ and Ψ satisfy the continuity condition $\psi(s_2) = (\Psi(s_2))_{2,2}$. The orthogonality conditions derived from this transformation are

$$\begin{aligned} \frac{dY_0}{ds} + \sum_\mu [T_\mu, Y_\mu] &= 0\,, \\ \frac{dy_0}{ds} &= 0\,, \end{aligned} \tag{2.46}$$

and $\mathcal{Y}_0$ must be continuous:

$$y_0 = (Y_0(s_2))_{2,2}\,. \tag{2.47}$$

It is noteworthy that gauge orthogonality imposes similar conditions on $\mathcal{Y}_0$ as those already imposed on $\mathcal{Y}_i$ by the linearised Nahm equations. This is related to the hyperkählerity of $\mathcal{M}_{(2,1)}$.

It is useful that the conditions above are similar to those found by Dancer [9]. The conditions on the Y_μ are precisely the same. The additional conditions do not impose additional constraints; they dictate the values of y_μ corresponding to a given Y_μ. This means that the tangent vectors at the point where (T_0, T_1, T_2, T_3) is $(0, f_1(s)e_1, f_2(s)e_2, f_3(s)e_3)$ are known; they follow from those presented in [9]. For the sake of completeness, these are reproduced here. $Y_\mu(s)$ are elements of the Lie algebra of SU(2) and so can be represented by $Y_\mu(s) = \sum_{i=1}^3 Y_{\mu i}(s)e_i$. Representing the index i as a vector index the solutions of the linearised Nahm equation are

$$\begin{aligned} Y_0(s) &= \begin{pmatrix} \dot f_1(s)I_4 \\ \dot f_2(s)I_3 + m_3/f_2(s) \\ -\dot f_3(s)I_2 - n_2/f_3(s) \end{pmatrix}, \qquad & Y_1(s) &= \begin{pmatrix} \dot f_1(s)I_1 \\ \dot f_2(s)I_2 + m_2/f_2(s) \\ \dot f_3(s)I_3 + n_3/f_3(s) \end{pmatrix}, \\ Y_2(s) &= \begin{pmatrix} -\dot f_1(s)I_2 \\ \dot f_2(s)I_1 + m_1/f_2(s) \\ -\dot f_3(s)I_4 - n_4/f_3(s) \end{pmatrix}, \qquad & Y_3(s) &= \begin{pmatrix} -\dot f_1(s)I_3 \\ \dot f_2(s)I_4 + m_4/f_2(s) \\ \dot f_3(s)I_1 + n_1/f_3(s) \end{pmatrix}, \end{aligned} \tag{2.48}$$

where

$$I_\lambda(s) = m_\lambda g_1(s) + n_\lambda g_2(s)\,, \tag{2.49}$$

and m_λ, n_λ are real numbers which parametrise the eight dimensional tangent space. g_1 and g_2 are the incomplete elliptic integrals

$$\begin{aligned} g_1(s) &= \int_{s_1}^{s} \frac{1}{f_2(s)^2} ds\,, \\ g_2(s) &= \int_{s_1}^{s} \frac{1}{f_3(s)^2} ds\,. \end{aligned} \tag{2.50}$$

The tangent vectors transform in a simple fashion under the SO(3) and SU(2) group actions. Tangent vectors at the general point in $\mathcal{M}^0_{(2,1)}$ given by (2.34) are

$$\{\mathcal{Y}_0, \mathcal{Y}_i\} = \left\{G\mathcal{Y}_0G^{-1}, GA_{ij}\mathcal{Y}_jG^{-1}\right\}. \tag{2.51}$$

There are eight independent tangent vectors at every point of $\mathcal{M}^0_{(2,1)}$. Given a set of eight coordinates α_a on $\mathcal{M}^0_{(2,1)}$, a basis of eight tangent vectors given in components as $\mathcal{Y}^a_\mu$ can be defined by

$$d\alpha_a(\mathcal{Y}^b) = \lim_{\epsilon\to 0}\left(\frac{\alpha_a(\mathcal{T}_\mu + \epsilon\mathcal{Y}^b_\mu) - \alpha_a(\mathcal{T}_\mu)}{\epsilon}\right) = \delta^b_a\,. \tag{2.52}$$

This method is then used to construct tangent vectors orthonormal to the one-forms.

2.7.1 The dual bases of one-forms and tangent vectors

In order to calculate the metric, we need a basis of one-forms and a basis of tangent vectors such that the two bases are dual in the sense of (2.52). We wish to replace the one-form ρ'_8 with a one-form which is derived from the well-defined angle α_8 rather then α'_8. However, if we proceed in the straightforward manner using the basis of one-forms derived from the coordinates ρ_a for $a = 1, \ldots, 7$ and

$$\rho'_8 + d\left(i\int_{s_2}^{s_3} t_0 ds\right) = d\alpha_8 + \cos\alpha_6 d\alpha_7\,, \tag{2.53}$$

we create unnecessary computational difficulties. The expressions for the tangent vectors which result are both very complicated and very different from those used by Dancer to calculate the $(2,[1])$ metric. Instead, we apply the orthonormalisation procedure as follows. We begin by defining the eight tangent vectors used by Dancer [9] in the calculation of the $(2,[1])$ metric. These are $\mathcal{Y}'^a$ for $a = 1, \ldots, 8$ which are dual to ρ_a for $a = 1, \ldots, 7$ and to ρ'_8. Thus

$$\rho_a(\mathcal{Y}'^b) = \delta^b_a\,, \tag{2.54}$$

for $a = 1, \ldots, 7$ and $b = 1, \ldots, 8$ and

$$\rho'_8(\mathcal{Y}'^b) = \delta^b_8\,, \tag{2.55}$$

for $b = 1, \ldots, 8$. As these are the $(2,[1])$ tangent vectors, the orthonormalisation depends on Y'^a and not on y'^a.

We must now introduce ρ_8, which is the modified form of ρ'_8. If we proceed in the straightforward manner illustrated in (2.53) and we use

$$d\left(i\int_{s_2}^{s_3} t_0 ds\right)(\mathcal{Y}'^a) = imy'^a_0\,, \tag{2.56}$$

we see that the one-form (2.53) is not dual to the convenient set of tangent vectors. Instead we define

$$\rho_8 = d\alpha_8 + \cos\alpha_6 d\alpha_7 - im\sum_{a=1}^{7} y_0^a \rho_a \,. \tag{2.57}$$

With this new eighth one-form $\rho_8(\mathcal{Y}'^a)$ is zero for $a = 1,\ldots,7$. Clearly, the conditions $\rho_a(\mathcal{Y}'^b) = \delta_a^b$ for $a, b = 1,\ldots,7$ are unaffected. Also, ρ_8 is a good one-form as it involves only the exterior derivatives of well-defined coordinates. It only remains to look at the action of ρ_8 on $\mathcal{Y}'^8$. This is

$$\rho_8(\mathcal{Y}'^8) = \Omega\,, \tag{2.58}$$

where

$$\Omega = 1 + imy_0'^8\,. \tag{2.59}$$

Thus, the basis of dual tangent vectors is given by $\mathcal{Y}^a$ where

$$\mathcal{Y}^a = \mathcal{Y}'^a\,, \tag{2.60}$$

for $a = 1,\ldots,7$ and

$$\mathcal{Y}^8 = \frac{1}{\Omega}\mathcal{Y}'^8\,. \tag{2.61}$$

What this means is that ρ_8 is proportional to $\rho_8' = d\alpha_8' + \cos\alpha_6 d\alpha_7$, in fact,

$$\rho_8 = \Omega\rho_8' = \Omega(d\alpha_8' + \cos\alpha_6 d\alpha_7)\,. \tag{2.62}$$

An explicit expression for Ω is given later. Note that, apart from this change to the eighth one-form and eighth tangent vector, the one-forms and tangent vectors are precisely those used in the construction of the $(2,[1])$ metric [9, 26]. The eight tangent vectors can be written out in terms of the coordinates m_λ and n_λ above but it would be tedious to give these expressions here.

In summary, the one-form ρ_8' is well-defined but is not identified modulo 2π under the group action and cannot therefore be written in terms of Euler angles. We have constructed a one-form ρ_8 in terms of well-defined coordinates which turns out to be simply a rescaling of ρ_8'.

2.8 The metric

The tangent vectors are uniquely determined by the orthonormalisation procedure in section 2.7 and we can now construct the metric on $\mathcal{M}^0_{(2,1)}$ in terms of the inner products of these tangent vectors. The contribution to the metric from the lefthand Nahm data is given by the formula, found in [9],

$$ds^2_{(2,\,)} = -\int_{s_1}^{s_2}\sum_{a,b=1}^{8}\sum_{\mu}\operatorname{trace} Y_\mu^a Y_\mu^b ds\,\rho_a\rho_b\,. \tag{2.63}$$

The contribution to the metric from the righthand Nahm data is given by a similar formula

$$ds^2_{(\ ,1)} = -\int_{s_2}^{s_3} \sum_{a,b=1}^{8} \sum_{\mu} y^a_\mu y^b_\mu ds\, \rho_a \rho_b = -\int_{s_2}^{s_3} \sum_{\mu} dt_\mu dt_\mu\,. \tag{2.64}$$

The explicit expression for the metric is given in the next section.

3. The explicit metric

To calculate the explicit metric, it is convenient to follow the version of the $m = 0$ metric calculation given in [26]. In $\mathcal{M}^0_{(2,[1])}$ the SU(2) group action and the SO(3) rotational action are both isometric. This allows the metric to be calculated at one point on the SU(2)×SO(3) orbit; there is a two-parameter space of such orbits. For the $(2,1)$ metric, the isometric actions on $\mathcal{M}^0_{(2,1)}$ are the U(1) action of $\mathcal{G}^{0,0,\star}/\mathcal{G}^{0,0,0}$ and the SO(3) rotational action. The righthand Nahm data, corresponding to a $(\ ,1)$-monopole at $\mathbf{r}$, break the SU(2) group isometry to a U(1) isometry. This means that the relative metric must be calculated on the whole four-dimensional space of SO(3)×U(1) orbits. This space is parameterised by two relative separations and two Euler angles giving the relative orientation of the monopoles. The isometry is used to calculate the metric for an infinitesimal SO(3)×U(1) action which will give the metric on the whole of $\mathcal{M}^0_{(2,1)}$.

Working at a point infinitesimally close to the identity on the SO(3)×U(1) orbit means that the coordinates α_6, α_7 and α_8' are equal to the Euler angles θ, χ and ϕ respectively, up to infinitesimal terms in the SO(3) action. However, when the derivative of one of these coordinates is taken, these extra terms will, in general, contribute non-trivial couplings of the SO(3) one-forms to $d\theta$, $d\chi$ and $d\phi$. This is shown explicitly in (3.3). Of course the gauge-dependent quantities α_8' and ϕ do not appear in the metric, although, as discussed in section 2.7, their exterior derivatives are well-defined one-forms on the moduli space and do appear in the metric.

3.1 The contribution from the lefthand Nahm data

The notation used in [9, 26] is adopted here. Much of the calculation involves evaluating the elliptic functions $f_i(s)$ at $s = s_2$ and so, in the rest of the paper, if there is no explicit argument given, it is assumed that the value at $s = s_2$ is used, that is $f_i = f_i(s_2)$. The same notation is used for the elliptic integrals (2.50): g_1 is used to mean $g_1(s_2)$ and g_2 to mean $g_2(s_2)$. Of course, these integrals are still incomplete elliptic integrals, even though they are integrals over the whole of $(s_1, s_2]$. It is also convenient to introduce the combinations

$$\begin{aligned} X &= f_1 f_2 f_3\,, \\ p_1 &= g_1 + \frac{1}{X}\,, \end{aligned}$$

$$
\begin{aligned}
p_2 &= g_2 + \frac{1}{X}\,, \\
p_3 &= g_1 + g_2 + \frac{1}{X}\,.
\end{aligned}
$$

In this notation, the scale Ω introduced earlier (2.58) is

$$
\Omega = 1 + \frac{m}{2}\left[\frac{X(g_1+g_2)}{f_1^2 p_3}\sin^2\theta\cos^2\chi + \frac{Xp_1}{g_1 f_2^2}\sin^2\theta\sin^2\chi + \frac{Xp_2}{g_2 f_3^2}\cos^2\theta\right]. \tag{3.1}
$$

It is the normalisation of the one-form ρ_8.

In terms of the elliptic function parameters D and k, the Euler angles (2.37) on the ellipsoid and the SO(3) one-forms $\{\sigma_i\}$ defined by

$$
A^\dagger dA|_{s=s_1} = \frac{i}{2}\sum_i \tau_i\sigma_i\,, \tag{3.2}
$$

explicit expressions for our set of one-forms are

$$
\begin{aligned}
\rho_1 &= d\alpha_1 = d(-k'^2D^2)\,, \\
\rho_2 &= d\alpha_2 = d(-D^2)\,, \\
\rho_3 &= d\alpha_3 = -k'^2D^2\sigma_3\,, \\
\rho_4 &= d\alpha_4 = -D^2\sigma_2\,, \\
\rho_5 &= d\alpha_5 = k^2D^2\sigma_1\,, \\
\rho_6 &= d\alpha_6 = d\theta + \frac{f_2}{f_3}\sin\chi\sigma_1 + \frac{f_1}{f_3}\cos\chi\sigma_2\,, \\
\rho_7 &= \sin\alpha_6 d\alpha_7 = \sin\theta d\chi + \frac{f_2}{f_3}\cos\theta\cos\chi\sigma_1 - \frac{f_1}{f_3}\cos\theta\sin\chi\sigma_2 + \frac{f_1}{f_2}\sin\theta\sigma_3\,, \\
\frac{1}{\Omega}\rho_8 &= d\alpha_8' + \cos\alpha_6 d\alpha_7 = d\phi + \cos\theta d\chi - \frac{f_2}{f_3}\sin\theta\cos\chi\sigma_1 + \\
&\qquad\qquad + \frac{f_1}{f_3}\sin\theta\sin\chi\sigma_2 + \frac{f_1}{f_2}\cos\theta\sigma_3\,.
\end{aligned} \tag{3.3}
$$

k' is the dual modulus: $k'^2 = 1 - k^2$. These expressions for the one-forms hold at a point on the SO(3)×U(1) orbit.

It is clear that any linearly-independent set of one-forms constructed from these could be suitable one-forms and could be used to describe the metric. The set of one-forms that lead to the simplest expression for the metric and most emphasise the connection to the $(2,[1])$ metric [26] are

$$
\begin{aligned}
\kappa_1 &= -\sin\chi\rho_6 - \cos\chi\cos\theta\rho_7 + \sin\theta\cos\chi\frac{1}{\Omega}\rho_8\,, \\
\kappa_2 &= -\cos\chi\rho_6 + \sin\chi\cos\theta\rho_7 - \sin\theta\sin\chi\frac{1}{\Omega}\rho_8\,, \\
\kappa_3 &= -\sin\theta\rho_7 - \cos\theta\frac{1}{\Omega}\rho_8\,.
\end{aligned} \tag{3.4}
$$

In terms of these, the contribution to the metric from the traceless part of the lefthand Nahm data can be written

$$
\begin{aligned}
ds^2_{(2,\,)} = & \frac{1}{8}\left[X(g_1 d\alpha_1 + g_2 d\alpha_2)^2 + g_1 d\alpha_1^2 + g_2 d\alpha_2^2\right] + \\
& + \frac{1}{2}\left(a_1\sigma_1^2 + a_2\sigma_2^2 + a_3\sigma_3^2\right) + \\
& + \frac{1}{2}\left[(b_1\sigma_1 + c_1\kappa_1)^2 + (b_2\sigma_2 + c_2\kappa_2)^2 + (b_3\sigma_3 + c_3\kappa_3)^2\right],
\end{aligned} \tag{3.5}
$$

where

$$
\begin{aligned}
a_1 &= \frac{g_1 g_2 k^4 D^4}{g_1 + g_2}, & a_2 &= \frac{g_2 p_3 D^4}{p_1}, & a_3 &= \frac{p_3 g_1 D^4 k'^4}{p_2}, \\
b_1 &= -k^2 D^2 \sqrt{\frac{g_2^2}{X(g_1 + g_2)p_3}}, & b_2 &= \frac{g_2 D^2}{\sqrt{X g_1 p_1}}, & b_3 &= \frac{g_1 D^2 k'^2}{\sqrt{X g_2 p_2}},
\end{aligned} \tag{3.6}
$$

and

$$
c_1 = \frac{1}{f_1}\sqrt{\frac{X(g_1 + g_2)}{p_3}}, \qquad c_2 = \frac{\sqrt{X g_1 p_1}}{f_2 g_1}, \qquad c_3 = \frac{\sqrt{X g_2 p_2}}{f_3 g_2}. \tag{3.7}
$$

3.2 The contribution from the righthand Nahm data

Next, let us turn to the contributions to the metric from the righthand Nahm data. Inevitably, these have rather long expressions as it is this part of the metric which breaks the isometry from SU(2) to U(1). In the limit of large separation, where the SU(2) isometry is restored, these expressions simplify. The contribution from t_1, t_2 and t_3 is

$$
ds^2 = -\int_{s_2}^{s_3} \sum_i \sum_{a,b=1}^{7} y_i^a y_i^b \rho_a \rho_b ds = m d\mathbf{r} \cdot d\mathbf{r}\,. \tag{3.8}
$$

These Nahm data are functions of only the spatial position coordinates α_a where $a = 1, \ldots, 7$ and, consequently, the metric contribution will only involve ρ_a where $a = 1, \ldots, 7$. The vector $\mathbf{r}$ is known explicitly, it is

$$
\mathbf{r} = \frac{1}{2} A \begin{pmatrix} -f_1 \sin\theta\cos\chi \\ f_2 \sin\theta \sin\chi \\ f_3\cos\theta \end{pmatrix}, \tag{3.9}
$$

where A is the SO(3) matrix A_{ij} defined in (2.23). Differentiating this expression gives the one-form $d\mathbf{r}$ as

$$
\begin{aligned}
2dr_1 = & \frac{1}{2} f_2 f_3 \sin\theta\cos\chi(g_1 d\alpha_1 + g_2 d\alpha_2) - \frac{1}{f_3}\cos\theta D^2 \sigma_2 + \\
& + \frac{1}{f_2}\sin\theta\sin\chi D^2 k'^2 \sigma_3 + f_1 \cos\theta \kappa_2 - f_1 \sin\chi \sin\theta \kappa_3,
\end{aligned}
$$

$$
\begin{aligned}
2dr_2 &= -\frac{1}{2}f_1 f_3 \sin\theta \sin\chi(p_1 d\alpha_1 + g_2 d\alpha_2) + \frac{1}{f_3}k^2 D^2 \cos\theta\sigma_1 - \\
&\quad -f_2 \cos\theta\kappa_1 - f_2 \cos\chi \sin\theta\kappa_3 , \\
2dr_3 &= -\frac{1}{2}f_1 f_2 \mathrm{cos}\theta(g_1 d\alpha_1 + p_2 d\alpha_2) + f_3 \sin\theta \sin\chi\kappa_1 + f_3 \sin\theta \cos\chi\kappa_2 . \qquad (3.10)
\end{aligned}
$$

The final term in the metric is associated with changes of the phase of the (, 1)-monopole. This is

$$
\frac{1}{m}\left[d(i\int_{s_2}^{s_3} t_0 ds)\right]^2 = -m \sum_{a,b=1}^{8} y_0^a y_0^b \rho_a \rho_b . \qquad (3.11)
$$

where

$$
\begin{aligned}
2i\sum_{a=1}^{8} y_0^a \rho_a &= -\frac{(g_1+g_2)X}{p_3 f_1^2}\sin\theta\cos\chi\kappa_1 + \frac{p_1 X}{g_1 f_2^2}\sin\theta\sin\chi\kappa_2 + \frac{p_2 X}{g_2 f_3^2}\cos\theta\kappa_3 + \qquad (3.12) \\
&\quad + \frac{g_2}{p_3 f_1}k^2 D^2 \sin\theta\cos\chi\sigma_1 + \frac{g_2}{g_1 f_2}D^2 \sin\theta\sin\chi\sigma_2 + \frac{g_1}{g_2 f_3}D^2 k'^2 \cos\theta\sigma_3 .
\end{aligned}
$$

3.3 The complete metric

For completeness, the whole metric is now displayed. It is

$$
\begin{aligned}
ds^2_{(2,1)} = &\frac{1}{8}\left[X(g_1 d\alpha_1 + g_2 d\alpha_2)^2 + g_1 d\alpha_1^2 + g_2 d\alpha_2^2\right] + \\
&+\frac{1}{2}\left(a_1\sigma_1^2 + a_2\sigma_2^2 + a_3\sigma_3^2\right) + \\
&+\frac{1}{2}\left[(b_1\sigma_1 + c_1\kappa_1)^2 + (b_2\sigma_2 + c_2\kappa_2)^2 + (b_3\sigma_3 + c_3\kappa_3)^2\right] + \\
&+\frac{m}{4}\left[\frac{1}{2}f_2 f_3 \sin\theta\cos\chi(g_1 d\alpha_1 + g_2 d\alpha_2) - \frac{1}{f_3}\cos\theta D^2\sigma_2 + \right. \\
&\qquad \left. + \frac{1}{f_2}\sin\theta\sin\chi D^2 k'^2\sigma_3 + f_1\cos\theta\kappa_2 - f_1\sin\chi\sin\theta\kappa_3\right]^2 + \\
&+\frac{m}{4}\left[-\frac{1}{2}f_1 f_3 \sin\theta\sin\chi(p_1 d\alpha_1 + g_2 d\alpha_2) + \right. \\
&\qquad \left. + \frac{1}{f_3}k^2 D^2\cos\theta\sigma_1 - f_2\cos\theta\kappa_1 - f_2\cos\chi\sin\theta\kappa_3\right]^2 + \\
&+\frac{m}{4}\left[-\frac{1}{2}f_1 f_2\cos\theta(g_1 d\alpha_1 + p_2 d\alpha_2) + f_3\sin\theta\sin\chi\kappa_1 + f_3\sin\theta\cos\chi\kappa_2\right]^2 + \\
&+\frac{m}{4}\left[\frac{g_2}{p_3 f_1}k^2 D^2\sin\theta\cos\chi\sigma_1 + \frac{g_2}{g_1 f_2}D^2\sin\theta\sin\chi\sigma_2 + \frac{g_1}{g_2 f_3}D^2 k'^2\cos\theta\sigma_3 - \right. \\
&\qquad \left. - \frac{(g_1+g_2)X}{p_3 f_1^2}\sin\theta\cos\chi\kappa_1 + \frac{p_1 X}{g_1 f_2^2}\sin\theta\sin\chi\kappa_2 + \frac{p_2 X}{g_2 f_3^2}\cos\theta\kappa_3\right]^2 \qquad (3.13)
\end{aligned}
$$

This is the metric on the space of traceless Nahm data. To derive the metric on $\mathcal{M}_{(2,1)}$, the mass m must be replaced by the reduced mass and the overall centre of mass term

$$(2M+m)\sum_{\mu} dR_{\mu}dR_{\mu} \tag{3.14}$$

must be added.

In the $m \to 0$ limit $\Omega = 1$ and $\{\kappa_i\}$ become a basis of body-fixed one-forms for the isometric SU(2) action on $\mathcal{M}_{(2,[1])}$. Bearing this in mind, it is easy to see that $ds^2_{(2,1)}$ reduces to $ds^2_{(2,[1])}$ when $m = 0$. Some of the properties of the metric are examined in section 5. The asymptotic metric which results when the (, 1)-monopole is a great distance from the (2,)-monopole is calculated in section 6.1.

4. The hyperkähler quotient construction and $\mathcal{M}_{(2,1)}$

The quotient of a hyperkähler manifold by a group action is not hyperkähler. In the hyperkähler quotient construction [20], the moment map is used to restrict the manifold to a level set whose quotient is hyperkähler. In short, if there is a free isometric group action of a compact group G on a hyperkähler manifold $\mathcal{M}$ which preserves the complex structures, there is a triplet of moment maps

$$\mu_i : \mathcal{M} \mapsto \mathfrak{g}^{\star}\,. \tag{4.1}$$

The contraction of the moment map μ_i with an element ξ of $\mathfrak{g}$ is a function on $\mathcal{M}$. The exterior derivative of this function is identical to the contraction of the Kähler form ω_i with the Killing vector field corresponding to ξ. The manifold

$$\mathcal{N} = \mu^{-1}(\mathbf{c})/G \tag{4.2}$$

is a hyperkähler manifold provided c_1 c_2 and c_3 are central elements of $\mathfrak{g}^{\star}$.

The Nahm equations appear naturally in the hyperkähler quotient construction because they are moment maps for the gauge action. The moduli space of Nahm data is the hyperkähler quotient of the space of general matrix functions with the correct boundary conditions by the gauge group [19]. This was exploited by Murray in his calculation of the $(1,1,\ldots,1)$ metric [35].

In this section, we are concerned with U(1) actions. This means that the moment map will be a map

$$\mu : \mathcal{M} \mapsto \mathbb{R}^3\,, \tag{4.3}$$

and the manifold $\mathcal{N}$ will have four fewer dimensions than $\mathcal{M}$.

The purpose of this section is to use the hyperkähler quotient construction to construct $\mathcal{M}_{(2,1)}$ and to derive some of its properties. Three U(1) actions are introduced below. Starting from a sixteen-dimensional manifold, described below, the uncentred $(2,1)$ moduli space will be obtained by a U(1) hyperkähler quotient. The centre of mass will then be fixed by a further U(1) hyperkähler quotient, leaving an eight-dimensional space. The final U(1) quotient will fix the position of the (, 1)-monopole, leaving a four-dimensional space. If it is fixed at the origin, this space is the Atiyah-Hitchin manifold. Physically, this corresponds to taking the limit of large (, 1)-monopole mass.

The U(1) actions are subgroups of the torus group whose elements are represented by

$$
\begin{aligned}
G &= \exp\left[i\frac{s-s_1}{s_2-s_1}\theta_1\mathbf{1}_2 - i\frac{s-s_1}{s_2-s_1}\theta_2\tau_3\right], \\
g &= \exp\left\{i(\theta_1+\theta_2) + i\frac{s-s_2}{s_3-s_2}[\theta_3-(\theta_1+\theta_2)]\right\}.
\end{aligned}
\tag{4.4}
$$

Roughly speaking, this $\mathrm{U}(1)^3$ contains a gauge U(1), the U(1) of overall phase and a U(1) of relative phase. The gauge U(1) has $\theta_1 = \theta_2$ and $\theta_3 = 0$.

We want to examine these U(1) actions on the moduli spaces. Coordinates on the orbit of the group are given by

$$
\begin{aligned}
\phi_1 &= i\int_{s_1}^{s_2} \mathrm{trace}\, T_0 ds\,, \\
\phi_2 &= -i\int_{s_1}^{s_2} \mathrm{trace}\, (T_0\tau_3) ds\,, \\
\phi_3 &= i\int_{s_2}^{s_3} t_0 ds
\end{aligned}
$$

and these transform as

$$
\begin{aligned}
\phi_1 &\to \phi_1 + 2\theta_1\,, \\
\phi_2 &\to \phi_2 + 2\theta_2\,, \\
\phi_3 &\to \phi_3 + \theta_3 - \theta_1 - \theta_2\,.
\end{aligned}
$$

In order to specify a U(1) subgroup of this $\mathrm{U}(1)^3$ we write for example

$$
\begin{aligned}
\theta_1 &= \alpha\theta\,, \\
\theta_2 &= \beta\theta\,, \\
\theta_3 - (\theta_1+\theta_2) &= \gamma\theta
\end{aligned}
$$

and then specify a $\mathbb{RP}^2$ vector (α,β,γ). The gauge transformation is given by $(1,1,-2)$. Let us write $\mathcal{S}(\alpha,\beta,\gamma)$ for the U(1) specified by (α,β,γ). The moment map for $\mathcal{S}(\alpha,\beta,\gamma)$ is a map

$$
\mu(\alpha,\beta,\gamma) : \mathcal{M} \to \mathbb{R}^3\,, \tag{4.5}
$$

and is

$$\mu_i(\mathcal{T}; \alpha, \beta, \gamma) = \alpha \mathrm{trace}\, T_i - \beta \mathrm{trace}\, (T_i \tau_3) + \gamma t_i\,. \tag{4.6}$$

Of course, the moment map only serves to define the level set. To calculate the hyperkähler manifold the level set must be quotiented by the U(1) action. The quotient by $\mathcal{S}(\alpha, \beta, \gamma)$ is performed, in effect, by introducing a group action fixing condition of the form

$$A\phi_1 + B\phi_2 + C\phi_3 = 0\,. \tag{4.7}$$

This fixing condition must be invariant under group transformations whose Killing vectors are orthogonal to the $\mathcal{S}(\alpha, \beta, \gamma)$ Killing vector $\mathcal{Y}(\alpha, \beta, \gamma)$ which is

$$\left(\frac{\alpha i}{M}\mathbf{1}_2 - \frac{\beta i}{M}\tau_3, \mathbf{0}_2, \mathbf{0}_2, \mathbf{0}_2\right), \tag{4.8}$$

on the lefthand interval, with $\mathbf{0}_2$ the zero matrix, and

$$\left(\frac{\gamma i}{m}, 0, 0, 0\right), \tag{4.9}$$

on the righthand interval. In the obvious notation

$$(\mathcal{Y}, \mathcal{Y}') = \frac{2}{M}(\alpha\alpha' + \beta\beta') + \frac{1}{m}\gamma\gamma'. \tag{4.10}$$

Furthermore,

$$\delta(A\phi_1 + B\phi_2 - C\phi_3) = -2\alpha' A - 2\beta' B - \gamma' C\,, \tag{4.11}$$

under $\mathcal{S}(\alpha', \beta', \gamma')$. Thus

$$(A, B, C) = \left(\frac{\alpha}{M}, \frac{\beta}{M}, \frac{\gamma}{m}\right) \tag{4.12}$$

is invariant under group transformations orthogonal to $\mathcal{S}(\alpha, \beta, \gamma)$. In terms of Nahm data the quotient condition is

$$\frac{\alpha}{M}\int_{s_1}^{s_2} \mathrm{trace}\, T_0 ds - \frac{\beta}{M}\int_{s_1}^{s_2} \mathrm{trace}\, (T_0\tau_3) ds + \frac{\gamma}{m}\int_{s_2}^{s_3} t_0 ds = 0\,. \tag{4.13}$$

This complements the moment map conditions.

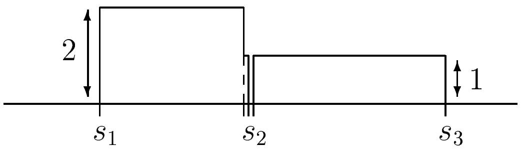
4.1 Hyperkähler quotient construction of $\mathcal{M}_{(2,1)}$

The moment map of the gauge action on $\mathcal{M}_{(2,1)}$ is

$$\mu_i(\mathcal{T}; 1, 1, -2) = \mathrm{trace}\, T_i - \mathrm{trace}\, (T_i\tau_3) - 2t_i\,. \tag{4.14}$$

and so the level set of the gauge transformation moment map $\mu^{-1}(\mathbf{0}; 1, 1, -2)$ satisfies the fixing condition $t_i = (T_i(s_2))_{2,2}$. What this means is that the moduli space $\mathcal{M}_{(2,1)}$ is the hyperkähler quotient of the space of unmatched Nahm data by this part of the gauge transformation. In this section, we use this hyperkähler quotient to attach a (, 1)-monopole to a (2,)-monopole. This gives an alternative derivation of the metric on $\mathcal{M}_{(2,1)}$.

We have seen how the lefthand Nahm data are the same, whether or not $m = 0$ and that in this sense the space of $(2,[1])$-monopoles can be interpreted as the space of $(2,\ )$-monopoles. We now consider the sixteen-dimensional manifold which will form the starting point for our series of $\mathrm{U}(1)$ hyperkähler quotients. This is the direct product of the space of uncentred $(2,[1])$-monopoles, with the moduli space of a 1-monopole. The moduli space of a 1-monopole is $\mathbb{R}^3 \times \mathrm{S}^1$. This describes the position and phase of the monopole. The Nahm data for this product are identical to the Nahm data for the space of $\mathrm{SU}(5)$ $(2,[1],[0],1)$-monopoles depicted below



(4.15)

where, as before, $M = s_2 - s_1$ and $m = s_3 - s_2$. There is a gap between the lefthand and righthand Nahm data to represent their independence from each other. In the $\mathrm{SU}(5)$ analogue, the gap represents the $(\ ,\ ,[0],\ )$ part of the charge.

The $(2,[1],\ ,\ )$-monopoles do not interact with the $(\ ,\ ,\ ,1)$-monopole. The sixteen dimensions of the moduli space of these monopoles separate into twelve moduli parameterising the space of uncentred $(2,[1])$-monopoles and four parameterising the position and phase of the 1-monopole. The Nahm data (T_0, T_1, T_2, T_3) at $s = s_2$ are $\mathfrak{u}(2)$-valued and the Nahm data (t_0, t_1, t_2, t_3) on the interval $[s_2, s_3]$ are a quadruplet of complex numbers. The metric on this space is

$$ds^2 = ds^2_{(2,[1])} + m d\mathbf{r}^2 + m dr_0^2\,, \tag{4.16}$$

where $\mathbf{r}$ is the position of the 1-monopole given by $\mathbf{r} = -i\mathbf{t}$ and $r_0 = \frac{i}{m}\int_{s_2}^{s_3} t_0 ds$.

The hyperkähler quotient of the sixteen-dimensional metric (4.16) by $\mathcal{S}(1,1,-2)$ is now performed. The three moment map equations are

$$t_i = (T_i(s_2))_{2,2}\,. \tag{4.17}$$

These are precisely what is required; they identified the position of the $(\ ,1)$-monopole with a point on the nonAbelian cloud of the $(2,[1])$-monopole. What remains is the calculation of the $(2,1)$-metric without including the interaction of the phases of the $(2,\ )$-monopole and $(\ ,1)$-monopole.

To complete the hyperkähler quotient construction, we must project the tangent vectors to be orthogonal to the tangent vector which generates the $\mathrm{U}(1)$ action. This amounts to removing from the metric the one-form which is preserved by the $\mathrm{U}(1)$ action. This is achieved by setting

$$\frac{1}{M}(\phi_1 + \phi_2) - \frac{2}{m}\phi_3 = 0\,. \tag{4.18}$$

The first two terms above combine into the one-form corresponding to change of phase of the nonAbelian cloud of the $(2,[1])$ monopole. The third term is the phase of the

(, 1)-monopole. Thus the U(1)-fixing condition (4.18) has a physical interpretation; it can be thought of as matching the phase angle of the cloud in the Dancer monopole with the phase angle of the (, 1)-monopole. Imposing this condition on the metric is equivalent to requiring that any tangent vector in the metric is orthogonal to the Killing vector $\mathcal{Y}(1,1,-2)$. Performing the hyperkähler quotient explicitly yields the metric (3.13).

4.2 Fixing the centre of mass using a hyperkähler quotient

In this sectionm we consider the torus action on the uncentred moduli space $\mathcal{M}_{(2,1)}$ derived above. The Nahm data are matched but not centred and so the moment map (4.6) is

$$\mu_i(\mathcal{T};\alpha,\beta,\gamma) = (\alpha-\beta)\mathrm{trace}\, T_i + (\gamma+2\beta)t_i\,. \tag{4.19}$$

Thereforem the centre of mass is fixed by a U(1) action with

$$\frac{\gamma+2\beta}{\alpha-\beta} = \frac{m}{M}\,. \tag{4.20}$$

Requiring that this action is orthogonal to $\mathcal{S}(1,1,-2)$ implies

$$\gamma = \frac{m}{M}(\alpha+\beta) \tag{4.21}$$

and hence the moment map is given by $(\alpha,\beta,\gamma) = (M,0,m)$. This means that the overall phase is set to zero, that is (A,B,C)=$(1,0,1)$. This defines $\mathcal{M}^0_{(2,1)}$ as a hyperkähler quotient of $\mathcal{M}_{(2,1)}$. It also demonstrates that the overall phase is $\phi_1+\phi_3$.

Equally well, the torus action could have been used to fix the traces of the lefthand Nahm data. The moment map is $\mu_i(\mathcal{T}) \propto \mathrm{trace} T_i$ if $\gamma+2\beta=0$. Thus, the U(1) action which sets $\mathrm{trace}\, T_i = 0$ and is orthogonal to gauge transformations is given by

$$(\alpha,\beta,\gamma) = \left(-\frac{2M+m}{m},1,-2\right). \tag{4.22}$$

The fixing condition in this case is given by $(A,B,C) = (-(2M+m)/m, 1, -2M/m)$. This seems at first sight to be unusual; the group action fixing condition is clarified by writing it as

$$-\frac{2M+m}{m}(\phi_1+\phi_3) + (\phi_2+\phi_3) = 0\,. \tag{4.23}$$

We see that the fixing condition is the sum of two gauge-invariant pieces. The first is the total phase of the $(2,1)$-monopole system and the second is the relative phases of the (2,)-monopole and the (, 1)-monopole.

The residual U(1), orthogonal both to the gauge U(1) and $\mathcal{S}(-(2M+m)/m,1,-2)$, is $\mathcal{S}(0,M,m)$. By (4.12) a coordinate on this U(1) is $\phi_2+\phi_3$. This coordinate is the coordinate α_8 adopted earlier. This can be verified by calculating ϕ_2 explicitly in terms of the group action parameters.

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In fact, any point along the line joining trace T_i and t_i can be fixed. However, all these manifolds are isometric up to a change in the $(\ , 1)$-monopole mass. The proof of this is essentially the same as the demonstration in section 2.4 that the $\mathcal{M}^0_{(2,1)}$ metric can be calculated from traceless Nahm data.

4.3 A hyperkähler quotient of $\mathcal{M}^0_{(2,1)}$

In [9], the hyperkähler quotient construction is used to construct a one-parameter family of four-dimensional hyperkähler manifolds from the moduli space $\mathcal{M}_{(2,[1])}$. In [22], it was noted that these manifolds can be thought of as an infinite mass limit in which the $(\ , 1)$-monopole in $\mathcal{M}_{(2,1)}$ becomes fixed in position. This is because the moment map fixes the $(\ , 1)$-monopole degrees of freedom. In this section, a similar hyperkähler quotient is performed on $\mathcal{M}_{(2,1)}$. It is found that the resulting four-dimensional manifolds are the same as those derived from $\mathcal{M}_{(2,[1])}$.

$\mathcal{M}^0_{(2,1)}$ has an isometric action of U(1) given by $\mathcal{S}(1, 1, 2m/M)$. The hyperkähler quotient of $\mathcal{M}^0_{(2,1)}$ results in a four-dimensional hyperkähler manifold

$$\mathcal{M}^4(\mathbf{y}; m) = \boldsymbol{\mu}^{-1}\left(\mathbf{y}, 1, 1, \frac{2m}{M}\right)/\mathrm{U}(1), \tag{4.24}$$

where $\mathbf{y} \in \mathbb{R}^3$. The mass of the $(\ , 1)$-monopole is a parameter for the moduli space $\mathcal{M}^0_{(2,1)}$ and so it might be expected that it also parameterises $\mathcal{M}^4(\mathbf{y}; m)$. For convenience, this possible dependence on m has been denoted explicitly. In fact, one of the results of this section is that $\mathcal{M}^4(\mathbf{y}; m)$ is actually independent of m.

The space of $(2, [1])$-monopoles is the $m \to 0$ limit of the space of $(2, 1)$-monopoles and so, in our notation, the hyperkähler manifolds constructed in [9] are denoted $\mathcal{M}^4(\mathbf{y}; 0)$. In fact, $\mathcal{M}^0_{(2,[1])}$ has a triholomorphic action of SU(2) and any U(1) subgroup of SU(2) can be used to perform the hyperkähler quotient. This appears to give a larger class of quotient spaces than there are for the $(2, 1)$ case. However, this is not the case because every U(1) subgroup results in the same quotient space.

Although $\mathbf{y}$ is an $\mathbb{R}^3$ vector, there is only a one-dimensional space of $\mathcal{M}^4(\mathbf{y}; m)$ for given m. This is because the action of SO(3) on $\mathcal{M}^0_{(2,1)}$ maps the space $\mathcal{M}^4(\mathbf{y}; m)$ isometrically to the space $\mathcal{M}^4(R\mathbf{y}; m)$, where $R \in \mathrm{SO}(3)$. This means that the three-parameter dependence of $\mathcal{M}^4(\mathbf{y}; m)$ on $\mathbf{y}$ reduces to a one-parameter dependence on $|\mathbf{y}|$.

Since the moment map for the $\mathcal{G}^{0,0,\star}/\mathcal{G}^{0,0,0}$ action on $\mathcal{M}^0_{(2,1)}$ is

$$\mu_i(\mathcal{T}) \propto t_i \tag{4.25}$$

the moment map fixes the position of the $(\ , 1)$-monopole, just as in the minimal symmetry breaking case.

The spaces $\mathcal{M}^4(\mathbf{y}; m)$ have an SO(2) isometry corresponding to the SO(2) subgroup of SO(3), which fixes the vector $\mathbf{y}$. If $\mathbf{y} = \mathbf{0}$, the whole SO(3) acts as an isometry. Since we have an expression for the metric on $\mathcal{M}_{(2,1)}$, it is possible to explicitly perform the quotient and derive the manifold $\mathcal{M}^4(\mathbf{0};m)$. The constraint $\mathbf{y}=\mathbf{0}$ implies the t_i are all zero. Substituting this into the full metric (3.13) gives the metric on the level set $\boldsymbol{\mu}^{-1}(\mathbf{0})$. Explicitly, $t_i=0$ is equivalent to

$$\begin{aligned} D &= \frac{1}{M}K(k)\,, \\ \theta &= \frac{\pi}{2}\,, \\ \chi &= 0\,, \end{aligned}$$

since $\mathrm{cn}_k K(k)=0$. Here and below, we will use the standard complete elliptic integrals $K\equiv K(k)$ and $E\equiv E(k)$. Substituting this into (3.13) gives

$$ds^2 = \frac{1}{4}\left(\frac{b^2}{K^2}dK^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2\right) + \frac{1}{2}\frac{EK}{M+mEK}\left(d\phi - \frac{k'K}{E}\sigma_1\right)^2\,, \quad (4.26)$$

where

$$\begin{aligned} a^2 &= \frac{2}{M}\frac{K(K-E)(E-k'^2K)}{E}\,, \\ b^2 &= \frac{2}{M}\frac{EK(K-E)}{E-k'^2K}\,, \\ c^2 &= \frac{2}{M}\frac{EK(E-k'^2K)}{K-E}\,. \end{aligned} \quad (4.27)$$

Now, to derive $\mathcal{M}^4(\mathbf{0};m)$ itself, the U(1) action must be quotiented out. This action corresponds to

$$\phi \to \phi + \phi_0\,, \quad (4.28)$$

and the U(1) is quotiented out by discarding the $(d\phi - k'K\sigma_1/E)^2$ term in (4.26). The resulting metric is,

$$ds^2 = \frac{1}{4}\left(\frac{b^2}{K^2}dK^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2\right)\,, \quad (4.29)$$

which is the Atiyah-Hitchin metric up to a scale of a quarter.

In [9], it was shown that when $\mathbf{y}=\mathbf{0}$ and $m=0$ the hyperkähler quotient gives the double cover of the Atiyah-Hitchin manifold. From the above, we see that the metric on $\mathcal{M}^4(\mathbf{0};m)$ is independent of m and $\mathcal{M}^4(\mathbf{0};m)$ is the double cover of the Atiyah-Hitchin manifold for all values of m. This is not surprising as $\mathcal{M}^4(\mathbf{0};m)$ is a four-dimensional hyperkähler manifold with an isometric SO(3) action and there is only a small number of manifolds with these properties. Furthermore, $\mathcal{M}^4(\mathbf{0};m)$ is the double cover of the Atiyah-Hitchin manifold when $m=0$. Therefore, if m is varied, the only possible variation of the metric is by an overall scaling. This point has also been made in [30].

The manifold $\boldsymbol{\mu}^{-1}(\mathbf{0})$ is a U(1) bundle over the double cover of the Atiyah-Hitchin manifold. The metric on $\boldsymbol{\mu}^{-1}(\mathbf{0})$ induced from the metric on $\mathcal{M}^0_{(2,1)}$ defines a U(1) connection on the bundle. The pullback of the curvature of the U(1) connection to the double cover of the Atiyah-Hitchin manifold is a closed two-form. In [11], this two-form was shown to be SO(3) invariant and anti-self-dual. It goes to minus itself under the transformation

$$(\sigma_1,\, \sigma_2,\, \sigma_3) \to (-\sigma_1,\, -\sigma_2,\, \sigma_3)\,. \tag{4.30}$$

Thus, it is the well known Sen form [16, 38]. This can be seen directly from (4.26) since the connection is $A = H\sigma_1$ with $H = -k'K/E$. The resulting curvature $F = dA$ is

$$F = Hd\sigma_1 + \frac{dH}{dK}dK \wedge \sigma_1\,, \tag{4.31}$$

and this is the Sen form.

It does not seem possible to perform the hyperkähler quotient tractably for $\mathbf{y} \neq \mathbf{0}$. However, it can be shown that the resulting manifold does not depend on m. From the Nahm construction, the only m dependence in the metric arises from the terms

$$m\sum_\mu x_\mu z_\mu\,, \tag{4.32}$$

in the inner product (2.43) of tangent vectors $\mathcal{X}$ and $\mathcal{Z}$. Since the moment map fixes t_i, this means that tangent vectors on $\boldsymbol{\mu}^{-1}(\mathbf{y})$, given by $\mathcal{X}_\mu = (X_\mu, x_\mu)$ have $x_i = 0$. Performing the U(1) quotient on $\boldsymbol{\mu}^{-1}(\mathbf{y})$ to give $\mathcal{M}^4(\mathbf{y}; m)$ amounts to projecting tangent vectors so that they are orthogonal to the tangent vector that generates this U(1) action. The tangent vector which generates this action has x_0 as its only nonzero component. This means that the orthogonal projection sets the x_0 component of that tangent vector to zero. Thus, there is no m dependence in $\mathcal{M}^4(\mathbf{y}; m)$. This supports the conjecture that there is only a one-parameter family of hyperkähler deformations of the Atiyah-Hitchin manifold.

5. Some properties of $\mathcal{M}_{(2,1)}$

In this section, we investigate some of the properties of the $(2,1)$ metric. In section 5.1, the rational map description of the moduli space is used to derive the topology of $\mathcal{M}_{(2,1)}$ and of $\mathcal{M}^0_{(2,1)}$. In section 5.2, the metric on the geodesic submanifold of axially symmetric $(2,1)$-monopoles is derived. By examining the behaviour of the monopoles on this geodesic submanifold, it is possible to infer something of $(2,1)$-monopole dynamics.

5.1 Rational maps

In this section, the rational map description of $\mathcal{M}_{(2,1)}$ is discussed. The space $\mathcal{M}_{(2,1)}$ is diffeomorphic to the space of based rational maps from $\mathbb{C}$ to the space of total flags in $\mathbb{C}^3$, denoted $F\mathbb{C}^3$. A total flag inside an n-dimensional vector space V_n is a series of vector subspaces $0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n$, where V_i has dimension i. Thus, an element of $F\mathbb{C}^3$ is a pair consisting of a complex plane and a complex line lying in that plane. A rational map from $\mathbb{C}$ to $F\mathbb{C}^3$ consists of a holomorphic map from $\mathbb{C}$ to a line in $\mathbb{C}^3$ and another holomorphic map from $\mathbb{C}$ to the plane in $\mathbb{C}^3$ containing the aforementioned line. These maps are not independent, since the image line of a point in $\mathbb{C}$ must lie in the image plane. However, each of the maps has a separate degree, these degrees are the topological charges of the corresponding SU(3) monopole.

This diffeomorphism was originally described as a diffeomorphism between the rational map space and the moduli space of Nahm data [24]. A more recent and more direct description of the diffeomorphism uses the Hitchin equation to construct the rational map directly from the monopole field [28].

The Hitchin equation is a scattering equation along a fixed oriented line v in $\mathbb{R}^3$,

$$(D_v - i\Phi)u = 0\,. \tag{5.1}$$

D_v is the covariant derivative in the v direction. If we choose v to be in the x_3 direction then there is a complex plane of lines parameterised by $z = x_1 + ix_2 \in \mathbb{C}$. The rational map is obtained by considering the solutions of the Hitchin equation (5.1) along these lines.

The Hitchin equation has three independent solutions. The asymptotic behaviour of Φ is known and substituting from (1.4) shows there is a spanning set of solutions u_1, u_2 and u_3 with the behaviour

$$\begin{aligned}
\lim_{x_3\to\infty} u_1(x_3;z)x_3^{-k_1/2}e^{s_1x_3} &= e_1\,,\\
\lim_{x_3\to\infty} u_2(x_3;z)x_3^{(k_1-k_2)/2}e^{s_2x_3} &= e_2\,,\\
\lim_{x_3\to\infty} u_3(x_3;z)x_3^{k_2/2}e^{s_3x_3} &= e_3\,,
\end{aligned} \tag{5.2}$$

where the e_i are the unit eigenvectors of $\lim\limits_{x_3\to\infty}\Phi$.

Thus, the $x_3 \to \infty$ asymptotic Hitchin equation nominates three particular solutions distinguished by their rate of growth or decay. The $x_3 \to -\infty$ asymptotic Hitchin equation also distinguishes three solutions. These are used to construct the flag. In the three-dimensional space of solutions, there is a one-dimensional subspace generated by the solution which decays at the fastest rate as $x_3 \to -\infty$ and a two-dimensional subspace spanned by this solution and the next fastest decaying solution. These subspaces can be written as linear combinations of the u_1, u_2 and

u_3 and the linear coefficients define a total flag in $\mathbb{C}^3$, that is, an element of $F\mathbb{C}^3$. The flag depends on z and the Bogomolny equations then imply that this flag varies holomorphically in z. The monopole charge determines the degree of the map. The monopole boundary conditions for large z show that the map is based. This means that as $z \to \infty$ the map approaches a fixed element in $F\mathbb{C}^3$.

Thus, we are interested in based rational maps of degree $(2,1)$. It is not difficult to write down the most general map of this type. Rather than describing the plane by a pair of holomorphic lines spanning it, we describe it by specifying a line in the plane and a line perpendicular to it. This perpendicular line is antiholomorphic. Thus, the general degree $(2,1)$ based rational map is

$$\begin{aligned} E_1 &= \left(1, \frac{az+b}{z^2+cz+d}, \frac{ez+f}{z^2+cz+d}\right), \\ E_2 &= \left(\frac{\alpha}{z-\gamma}, \frac{\beta}{z-\gamma}, 1\right), \end{aligned} \tag{5.3}$$

where E_1 describes a line in $\mathbb{C}^3$ and $E_2^\star$ describes the line perpendicular to a plane in $\mathbb{C}^3$. $E_2^\star$ is the complex conjugate of E_2. For E_1 to lie in the plane defined by $E_2^\star$ the -hermitean inner product of E_1 and $E_2^\star$ must vanish.

If $af - be = 0$, then it follows from the orthogonality condition that the map is not genuinely of degree $(2,1)$. Either E_1 is not a genuine degree two map, because $z = -b/a$ is a root of $z^2 + cz + d$ as well as of $az + b$ and $ez + f$, or E_2 is degree zero because $\alpha = \beta = 0$. This means that $af - be \neq 0$ and orthogonality determines α, β and γ in terms of a, b, c, d, e and f. In other words, E_2 is determined from E_1. This means that the space of degree $(2,1)$ rational maps is isomorphic to $(a,b,c,d,e,f) \in \mathbb{C}^6$ with the constraint $af - be \neq 0$. This is $\mathrm{GL}(2,\mathbb{C}) \times \mathbb{C}^2$. Thus, the moduli space is topologically equivalent to $\mathrm{GL}(2,\mathbb{C}) \times \mathbb{C}^2$ or, equivalently, $\mathrm{U}(2)\times\mathbb{R}^8$.

The manifold $\mathcal{M}_{(2,1)}$ has the product form

$$\mathcal{M}_{(2,1)} = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}^0_{(2,1)}}{\mathbb{Z}}. \tag{5.4}$$

$\mathcal{M}^0_{(2,1)}$ is the relative moduli space obtained by quotienting out the centre of mass action, $\mathbb{R}^3 \times \mathbb{R}$. Using (5.1), it is possible to determine the effect of translations and gauge transforms on the map. A translation in the (x_1, x_2) plane by $w \in \mathbb{C}$ acts as

$$\begin{aligned} E_1(z) &\to E_1(z-w), \\ E_2(z) &\to E_2(z-w). \end{aligned} \tag{5.5}$$

This action can be used to set $c = 0$.

A translation in the x_3 direction by λ and a gauge transform $g = e^{\theta\Phi}$ has the following effect:

$$E_1(z) \to \left(1, \frac{e^{M(\lambda+i\theta)}(az+b)}{z^2+cz+d}, \frac{e^{(M+m)(\lambda+i\theta)}(ez+f)}{z^2+cz+d}\right). \tag{5.6}$$

The transformation on E_2 is determined by the transformation on E_1. The gauge transformation g changes the overall phase of the monopole. Notice that θ is an angle only when m/M is rational. Under this transformation

$$af - be \to e^{(2M+m)(\lambda+i\theta)}(af - be)\,, \tag{5.7}$$

and so $af - be$ can be set to one. Setting c to zero and $af - be$ to one fixes the centre of mass and overall phase, allowing us to conclude $\mathcal{M}^0_{(2,1)}$ is determined by $(a, b, d, e, f) \in \mathbb{C}^5$ such that $af - be = 1$. This implies that $\mathcal{M}^0_{(2,1)}$ is topologically equivalent to $\mathrm{SU}(2)\times\mathbb{R}^5$. The $\mathbb{Z}$ quotient in the product form (5.4) follows from the identification

$$\begin{aligned}
\theta &= \theta + \frac{2\pi n}{2M+m}\,,\\
(a\,,\,b) &= e^{-2\pi ni\frac{M}{2M+m}}\,(a\,,\,b)\,,\\
(e\,,\,f) &= e^{-2\pi ni\frac{M+m}{2M+m}}\,(e\,,\,f)\,,
\end{aligned} \tag{5.8}$$

where $n \in \mathbb{Z}$.

We remark that this agrees with the results in [9, 11] for the minimal symmetry breaking case. This is to be expected since $\mathcal{M}^0_{(2,[1])}$ is the smooth $m \to 0$ limit of $\mathcal{M}^0_{(2,1)}$ and the topological properties should be identical, as indeed they are. The manifold is given in this case by

$$\mathcal{M}_{(2,[1])} = \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}^0_{(2,[1])}}{\mathbb{Z}_2}\,, \tag{5.9}$$

where $\mathcal{M}^0_{2,[1]}$ is the moduli space of centred $(2,[1])$-monopoles. Since $m = 0$ here, the total phase is periodic and the $\mathbb{Z}$ quotient reduces to a $\mathbb{Z}_2$ quotient. Here, $\mathcal{M}_{(2,[1])}$ is the double cover of what is denoted M^8 in [9]. The rational map description of $\mathcal{M}_{(2,[1])}$ is given by maps from $\mathbb{C}$ to $\mathbb{CP}^2$, whose images do not lie in a $\mathbb{CP}^1$. This means that the rational map is given by E_1. E_1 does not lie in a $\mathbb{CP}^1$ if $af - be \neq 0$ and so this constraint applies for $(2,[1])$, as well as for $(2,1)$. In case of $(2,1)$, we have E_2 in addition to E_1 but E_2 is determined by E_1.

5.2 Geodesic submanifolds

The analysis of geodesics on monopole moduli spaces is a formidable task. However, the existence of tractable geodesic submanifolds allows one to partially deduce their behaviour. In this section, we locate the geodesic submanifold of axially symmetric $(2,1)$-monopoles Since the metric on $\mathcal{M}^0_{(2,1)}$ is now known, we can find the induced metric on the submanifold and use this to study the behaviour of axially symmetric monopoles.

The fixed point set of a isometric action on a Riemannian manifold is a totally geodesic submanifold. The geodesics on the submanifold are also geodesics on the manifold itself. This allows us to study $(2,1)$ geodesic behaviour using symmetry.

There is an isometric SO(3) action on $\mathcal{M}^0_{(2,1)}$ so the fixed point set of any subgroup of SO(3) will be a geodesic submanifold of $\mathcal{M}^0_{(2,1)}$ consisting of monopoles invariant under the subgroup. A compensating U(1) transform in $\mathcal{G}^{0,0,\star}/\mathcal{G}^{0,0,0}$ may be needed to keep the monopole invariant. This action is also isometric and if the combined transformations are a subgroup of SO(3)×U(1), then the fixed point set will still be a geodesic submanifold. The SO(3) and U(1) actions are defined on the the Nahm data. Symmetric Nahm data need not be strictly invariant under the relevant group action; it will generally change by a gauge transformation. To be precise about this, we need to check that the coordinates that describe a point on $\mathcal{M}^0_{(2,1)}$ are invariant under the group action. Such points, if any, then comprise the geodesic submanifold. Given a subgroup of SO(3), we look for Nahm data such that the eight coordinates: α_a, $a = 1, \ldots, 8$ of section 2.6 are invariant under a combined subgroup of SO(3)×U(1). This will define a geodesic submanifold.

5.2.1 Spherical symmetry

The only lefthand Nahm data T_i invariant under the SO(3) action, (2.23), has the form

$$T_i = -\frac{1}{(s - s_1)} e_i \,, \tag{5.10}$$

that is $D = k = 0$. From (3.9) the (, 1)-monopole will be positioned at

$$-it_i = \frac{1}{2}\left(0\,,\,0\,,\frac{-1}{M}\right)_i \,. \tag{5.11}$$

However, spherically symmetric Nahm data must have $t_i = 0$. This means that there is no spherically symmetric $(2, 1)$-monopole. In the $m \to 0$ limit, a spherically symmetric monopole appears. This has been known for some time [3].

5.2.2 Axial symmetry

Monopoles which are axially symmetric about the x_3-axis can be obtained by requiring $f_1(s) = f_2(s)$ in the ansatz (2.31). We must also require $t_1 = t_2 = 0$. This is achieved if $k = 1$ and θ is set to zero or π in (3.9). Setting $k = 1$ in the Euler top functions gives

$$\begin{aligned} f_1(s) = f_2(s) &= -D\mathrm{cosech}\left(D(s - s_1)\right), \\ f_3(s) &= -D\coth\left(D(s - s_1)\right), \end{aligned} \tag{5.12}$$

where, for convenience, we have set $M = s_2 - s_1 = 1$. The (, 1)-monopole is positioned at $(0,\, 0,\, \mp\frac{1}{2}D\coth D)$ with the sign depending on whether θ is zero or π. This is very similar to the axially symmetric Nahm data constructed in [9]. The $(2, [1])$ Nahm data are axially symmetric, if the action in (2.23) with $G(s_1) = e^{i\alpha\tau_3/2}$ is combined with an action $G' \in \mathcal{G}^{0,\star}$ that has $G'(s_2) = e^{i\alpha\tau_3/2}$. This means that a transformation of this form will leave α_1 to α_7 unchanged since these are coordinates when $m = 0$. Requiring the relative phase (2.39) to be unchanged determines what the compensating $\mathcal{G}^{0,0,\star}/\mathcal{G}^{0,0,0}$ transformation must be. In fact, it is easy to see by inspection that α_8 will be invariant if t_0 is unchanged by the combined transformation. This implies that $g(s_3) = e^{-i\alpha/2}$ since $(G'(s_2))_{2,2} = e^{-i\alpha/2}$. Thus the (2, 1) Nahm data are invariant under a combined transform of (2.23) and $\mathcal{G}^{0,0,\star}$ with $G(s_1) = e^{i\alpha\tau_3/2}$ and $g(s_3) = e^{-i\alpha/2}$. The combined transform is thus a diagonal U(1) subgroup of SO(3)×U(1) and the Nahm data are fixed under this action. D can take any positive value.

These Nahm data can be interpreted in terms of monopole configurations. The (, 1)-monopole position is $-it_i$. We will assume that the energy density of the (2,)-monopole configuration does not change significantly with changes of m, the mass of the (, 1)-monopole. This means that we assume the configuration is similar to the corresponding configuration of (2, [1])-monopoles. Thus, we follow [13] and interpret D as the separation of the (2,)-monopoles (in the asymptotic metric calculations of section 6.2, D is used with success as a separation parameter). We also assume $k = 0$ represents toroidal (2,)-monopoles [13]. For large D, these Nahm data represent two (1,)-monopoles separated along the x_3-axis with approximate positions $(0,\ 0,\ \pm\frac{1}{2}D)$. Since the (, 1)-monopole is positioned at $(0, 0, \frac{1}{2}D\coth D)$ or $(0, 0, -\frac{1}{2}D\coth D)$ it is close to one of the (1,)-monopoles. When it is very close, the configuration should look like a spherically symmetric (1, 1)-monopole well separated from a (1,)-monopole.

These Nahm data remain axially symmetric if acted on by the U(1) group $\mathcal{G}^{0,0,\star}/\mathcal{G}^{0,0,0}$ whose action commutes with the SO(3) action. Thus, there is a two-parameter family of axially symmetric monopoles. It is the U(1) orbit of the one-dimensional family parameterised by $D \in [0, \infty)$. We call this family the hyperbolic region because its Nahm data are hyperbolic. The induced metric on the hyperbolic region is easily obtained from (3.13) by substituting the constraints above:

$$ds^2 = \eta_1(D)dD^2 + \frac{\zeta_1(D)}{1 + m\zeta_1(D)}d\phi^2\,, \tag{5.13}$$

where η_1 and ζ_1 are given by

$$\begin{aligned}\eta_1(D) &= \frac{1}{2}(\sinh D\cosh D - D)(D - \tanh D)\frac{\cosh D}{D\sinh^3 D} + \\ &\quad + \frac{m}{4}\frac{(\sinh D\,\cosh D - D)^2}{\sinh^4 D}, \\ \zeta_1(D) &= \frac{1}{2}\frac{D(D - \sinh D\,\cosh D)}{\cosh D\,\sinh D(\tanh D - D)}\,.\end{aligned} \tag{5.14}$$

The hyperbolic region is only part of the geodesic submanifold of axially symmetric Nahm data. There is also a trigonometric region, in which

$$\begin{aligned}f_1(s) = f_2(s) &= -D\mathrm{cosec}\,(D(s - s_1))\,, \\ f_3(s) &= -D\cot\,(D(s - s_1))\,.\end{aligned} \tag{5.15}$$

By an argument which is identical to the above, these Nahm data are invariant under the diagonal subgroup of SO(3)×U(1). They correspond to $k = 0$ Euler top functions. However, if k is set to zero in (2.33), the Nahm data which result are axially symmetric about the x_1-axis. The solutions (5.15) correspond to a different ordering of the f_1, f_2 and f_3 and may be obtained from the ansatz space by rotation. For $t_1 = t_2 = 0$, θ is again zero or π and the position of the (, 1)-monopole is given by $(0, 0, \pm\frac{1}{2}D \cot D)$. This is the trigonometric region. $D \in [0, \pi)$ and the two regions are joined at $D = 0$, where the Nahm data are rational and k is not determined.

In the trigonometric region, the (2,)-monopoles are coincident and toroidal in shape. The trigonometric Nahm data also remain axially symmetric if acted on by the U(1) factor $\mathcal{G}^{0,0,\star}/\mathcal{G}^{0,0,0}$. The metric on the trigonometric region is

$$ds^2 = \eta_2(D)dD^2 + \frac{\zeta_2(D)}{1 + m\zeta_2(D)}d\phi^2 \tag{5.16}$$

with η_2 and ζ_2 given by

$$\begin{aligned}\eta_2(D) &= \frac{(\sin D\,\cos D - D)^2}{\sin^4 D}\left(\frac{1}{2}\frac{\sin D\,\cos D(D - \tan D)}{D\sin D\,\cos D - D^2} + \frac{m}{4}\right), \\ \zeta_2(D) &= \frac{1}{2}\frac{D(D - \sin D\,\cos D)}{\cos D\,\sin D(\tan D - D)}.\end{aligned} \tag{5.17}$$

The two regions fit together smoothly at $D = 0$ and together form a geodesic submanifold of $\mathcal{M}^0_{(2,1)}$. In the hyperbolic region, ζ_1 continually decreases as D increases and approaches $1/2$ as $D \to \infty$. Thus, the hyperbolic region asymptotes to a cylinder of radius $1/(2+m)$ as $D \to \infty$. In the trigonometric region, ζ_2 continually increases as D increases with $\zeta_2 \to \infty$ as $D \to \pi$. Thus, the trigonometric region asymptotes to a cylinder of radius $1/m$ as $D \to \pi$. The whole submanifold can be pictured as a surface which asymptotes at either end to cylinders of radii $1/(2 + m)$ and $1/m$ respectively. In the $m = 0$ limit it asymptotes to a cone at one end: this was studied in [12]. The surfaces for differing values of m are depicted in figure 2.

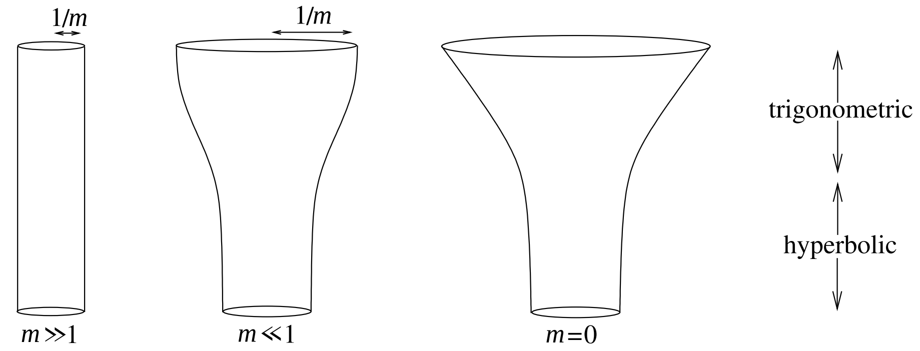


Figure 2: The axially symmetric moduli space for differing values of m.

Since there is a U(1) isometry, the geodesics are easy to analyse. The charge

$$\mathcal{Q} = \frac{2\zeta\dot{\phi}}{1+m\zeta}\,, \tag{5.18}$$

is conserved. Here $\dot{\phi} = d\phi/d\tau$, where τ is a parameter along the geodesic and we denote by ζ either ζ_1 or ζ_2 depending on which region the geodesic is in. Since the centre of mass and the phase of the (2,)-monopole are fixed, $\mathcal{Q}$ is the total electric charge of the (, 1)-monopole.

The following scattering process occurs. Starting near the boundary in the trigonometric region, that is the upper part of the surfaces in figure 2, D is close to π. The (2,)-monopole is at the origin and its fields look like those of the axially symmetric embedded SU(2) two-monopole. The (, 1)-monopole is approaching the (2,)-monopole from a large distance along the positive x_3-axis. Since the radius of the surface is continually decreasing, it is possible that the geodesic will only travel a certain distance downwards before returning upwards again. We can rewrite (5.13) and (5.16) in the form

$$ds^2 = \frac{1}{2}dx^2 + \frac{\zeta}{1+m\zeta}d\phi^2\,, \tag{5.19}$$

where

$$\left(\frac{dx}{dD}\right)^2 = 2\eta\,, \tag{5.20}$$

and x increases as the radius of the surface decreases. There are two conserved quantities on each geodesic, the electric charge $\mathcal{Q}$ and the energy $\mathcal{E}$. We can write $\mathcal{E}$ as

$$\mathcal{E} = \frac{1}{2}\dot{x}^2 + \frac{(1+m\zeta)}{4\zeta}\mathcal{Q}^2\,. \tag{5.21}$$

The geodesic returns if

$$m + \frac{1}{\zeta(x_0)} = \frac{4\mathcal{E}}{\mathcal{Q}^2}\,, \tag{5.22}$$

for some x_0. Holding $\mathcal{E}$ fixed and increasing $\mathcal{Q}$ decrease x_0, the point where the geodesic returns.

If the electric charge $\mathcal{Q} = 0$, then the geodesic never returns; x keeps increasing and the (, 1)-monopole passes through the (2,)-monopole configuration and on to the negative x_3-axis. The geodesic passes from the trigonometric region into the hyperbolic region and the toroidal (2,)-monopole breaks apart: as $D \to \infty$ the (, 1)-monopole is approximately positioned at $(0, 0, -D/2)$ and the (2,)-monopole configuration becomes particle-like with the (1,)-monopoles positioned at approximately $(0, 0, \pm D/2)$. The (, 1)-monopole is asymptotically coincident with one of the (2,)-monopoles, giving one spherically symmetric $(1, 1)$-monopole and one (1,)-monopole.

It is instructive to compare the $\mathcal{Q} = 0$ geodesic with the corresponding geodesic in the $\mathcal{M}^0_{(2,[1])}$. In that case $m = 0$, and the $(\ , 1)$-monopole is replaced by a cloud. Starting in the hyperbolic region, the geodesic describes two widely separated monopoles approaching each other along the x_3-axis. The cloud size is minimal. The monopole instantaneously forms the spherically symmetric $(2, [1])$-monopole and then the monopole deforms to a toroidal shape as it approaches the SU(2) embedded solution. As it does so, the cloud size continually increases. This is consistent with the $m \neq 0$ discussion if one adopts the view that the cloud size is a measure of the distance of the notional massless monopole position from the position of the massive monopoles. When the $(\ , 1)$-monopole is coincident with one of the $(2, \ )$-monopoles there is no cloud. The cloud appears only when the $(\ , 1)$-monopole is well separated from both of the $(2, \ )$-monopoles.

It is also instructive to compare the behaviour of $(1, 1)$-monopoles with the behaviour of axially symmetric $(2, 1)$-monopoles. In the axially symmetric submanifold a geodesic with nonzero $\mathcal{Q}$ may return and so the electric charge gives a repulsive interaction between the two different types of monopoles. For large $\mathcal{Q}$, the $(\ , 1)$-monopole approaches the toroidal $(2, \ )$-monopole configuration but the monopoles slow down and generically they stop and separate again. This behaviour agrees with that found in [8] in the dynamics of $(1, 1)$-monopoles. There, the geodesics were found to be hyperbolae; no bound geodesics exist. The relative electric charge of the two different types of monopoles has a repulsive effect. This leads us to conclude that the interaction of the two different types of monopoles is generally repulsive.

6. Asymptotic metrics

In this section, we derive two asymptotic expressions for the metric. In section 6.1, we consider the approximate simplification which occurs when the $(\ , 1)$-monopole separation is large. This is useful, because the general behaviour of the geodesics is very complicated. However, if the monopoles of different type are repulsive, the generic geodesic will correspond asymptotically to a $(\ , 1)$-monopole well separated from the $(2, \ )$-monopole configuration. Here the interaction is easy to understand; the $(2, \ )$-monopoles will interact like SU(2) monopoles but with a slight modification because of the distant $(\ , 1)$-monopole. This $(\ , 1)$-monopole interacts with the $(2, \ )$-monopole in a Taub-NUT like manner. Indeed, for large separations of the $(\ , 1)$-monopole and the $(2, \ )$-monopole the metric approximately simplifies to a direct product of the Atiyah-Hitchin metric and the Taub-NUT metric. The corrections to this simplification are algebraic in the $(\ , 1)$-monopole separation distance.

The asymptotic form of the metric which corresponds to large separation of the two $(1, \ )$-monopoles can be calculated by approximating the monopoles by point dyons using the methods of [34, 15, 33]. It has also been calculated by Bielawski [5] using Nahm data. As with the Taub-NUT approximation to the Atiyah-Hitchin metric, the point dyon metric has only exponentially small corrections. At first glance, it seems hard to imagine how the point dyon metric could be calculated as an approximation to the $(2,1)$ metric as presented above and so in section 6.2 we have calculated the radial terms of the point dyon metric from the $d\alpha_1$ and $d\alpha_2$ terms of the $(2,1)$ metric.

6.1 Large (, 1)-monopole separation

In this section, the (, 1)-monopole separation is taken to be large and the resulting approximate simplification to the metric is derived. The calculation is a generalisation of the $(2,[1])$ calculation in [26]; where it is shown that if the cloud size is large, the $\mathcal{M}^0_{(2,[1])}$ metric is approximately the direct product of the Atiyah-Hitchin metric and the flat $\mathbb{R}^4$ metric.

The (, 1)-monopole separation is large when the f_i are large. This means that DM must be close to $2K$. To leading order in $2K-DM$, $f_1=2r$ and $f_2=f_3=-2r$ where

$$2r \approx \frac{D}{2K-DM}\,. \tag{6.1}$$

This formula allows us to write $d\alpha_1$ and $d\alpha_2$ in terms of dK and dr. The incomplete elliptic integrals g_1 and g_2 can be approximated by complete integrals giving

$$\begin{aligned} g_1 &\approx \frac{2}{D^3k^2k'^2}(E-k'^2K) \\ g_2 &\approx \frac{2}{D^3k^2}(K-E)\,. \end{aligned} \tag{6.2}$$

Furthermore, since r is large X is large and $p_1\approx g_1$, $p_2\approx g_2$ and $p_3\approx g_1+g_2$ and so $\Omega\approx 1+mr$. Using these formula working out the approximate metric is just a matter of lengthy calculation and substitution. The approximate metric is

$$\frac{b^2}{K^2}dK^2+a^2\sigma_1^2+b^2\sigma_2^2+c^2\sigma_3^2+\left(m+\frac{1}{r}\right)\left[dr^2+r^2(\hat\sigma_1^2+\hat\sigma_2^2)\right]+\frac{r}{1+mr}\hat\sigma_3^2\,, \tag{6.3}$$

where the a^2, b^2 and c^2 are the Atiyah-Hitchin functions defined above (4.27) and the $\hat\sigma_i$ are

$$\begin{aligned} \hat\sigma_1 &= d\theta+\sin\chi\sigma_1-\cos\chi\sigma_2\,, \\ \hat\sigma_2 &= \sin\theta d\chi+\cos\theta\cos\chi\sigma_1+\cos\theta\sin\chi\sigma_2-\sin\theta\sigma_3\,, \\ \hat\sigma_3 &= d\phi+\cos\theta d\chi-\sin\theta\cos\chi\sigma_1-\sin\theta\sin\chi\sigma_2-\cos\theta\sigma_3\,. \end{aligned} \tag{6.4}$$

The $\hat\sigma_i$ thus contain invariant one-forms for both the SU(2) group action and the induced rotational action.

This is a satisfying result. The metric splits into two parts. One part describes the interaction of the (2,)-monopole. It interacts as if it was a 2-monopole. The other part describes the interaction of the (, 1)-monopole with the (2,)-monopole. It is just a Taub-NUT metric and describes the point dyon interaction of two distinct monopoles. It should be emphasised that there are order r^{-2} corrections to this approximate metric.

6.2 The point dyon metric

It was pointed out by Atiyah and Hitchin [1] that for large separation of the two monopoles, their metric is approximated with exponential accuracy by a singular Taub-NUT metric. It was subsequently demonstrated [34] that this asymptotic metric could also have been derived by examining the interactions of point sources of the fields. This point dyon method was generalised to SU(2) multimonopoles [15] and in [33] to larger groups. Although [33] is mostly concerned with (1,1,...,1)-monopoles, it describes what the point dyon metric is for any monopole. The point dyon metrics are discussed in a rigorous way by Bielawski [4, 5].

The radial part of this dyonic metric is

$$\left(m+\frac{1}{2r_1}+\frac{1}{2r_2}\right)dr^2+\left(\frac{1}{r_2}-\frac{1}{r_1}\right)drdR+\left(2M-\frac{2}{R}+\frac{1}{2r_1}+\frac{1}{2r_2}\right)dR^2\,, \quad (6.5)$$

where R is the distance of each of the (1,)-monopoles from the origin, r is the distance of the (,1)-monopole from the origin and r_1 and r_2 are the distances of each of the (1,)-monopoles from the (,1)-monopole. R is large. In this section, we derive this metric from the $d\alpha_1$ and $d\alpha_2$ terms of the $(2,1)$ metric to exponential accuracy in R. To do this we choose a specific configuration and only allow r and R to vary. This configuration is illustrated in figure 3. In this configuration

$$\begin{aligned} r_1 &= r-R, \\ r_2 &= r+R\,. \end{aligned} \quad (6.6)$$

We have chosen a region where r is bigger than R and the three monopoles are collinear. This is a convenient choice; however, the point dyon metric does not rely on r being large [5].

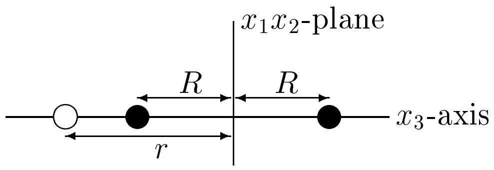


Figure 3: The monopole configuration for the point dyon metric calculation, the solid dots represent (1,)-monopoles, the other dot is the (,1)-monopole.

The separations r and R must be related to quantities appearing in the metric. This is easy for r, since it is well defined in the full metric, it is

$$r=-\frac{f_3}{2}\,. \quad (6.7)$$

Common sense, comparison with the Atiyah-Hitchin case and the numerical calculations of Dancer and Leese [13] all suggest that, up to exponential corrections,

$$R\approx\frac{D}{2}\,. \quad (6.8)$$

That this suggestion is correct is demonstrated by the success of the calculation. If R is large, D is large. Since $DM < 2K$, this implies $2K$ is large. For large K,

$$K \approx \log \frac{4}{k'} \tag{6.9}$$

and so k' is exponentially small [6, eq.600.05]. The rest of this section is concerned with approximating the metric to leading order in k'.

In order to evaluate the two integrals g_1 and g_2 we must evaluate the incomplete elliptic integral of the second kind, $E(u;k)$, for $u = DM$. It follows from the quasiperiodicity of $E(u;k)$, [6, eqs.113.02 & 903.01], that to leading order in k'

$$\begin{aligned} E(DM;k) &\approx 2 - \operatorname{sn}_k DM \\ &= 2 + \frac{D}{f_3} \\ &\approx 2 - \frac{R}{r} . \end{aligned} \tag{6.10}$$

Now g_1 can be evaluated. In Glaisher's notation, see for example [6, eq.120.02],

$$g_1 = \frac{1}{D^3} \int_0^{DM} \operatorname{sd}_k u \, du . \tag{6.11}$$

Now [6, eq.318.02]

$$\int \operatorname{sd}_k u \, du = \frac{1}{k^2 k'^2} \left[E(u;k) - k'^2 u - k^2 \operatorname{sn}_k u \operatorname{cd}_k u \right] \tag{6.12}$$

and the Jacobi functions at $u = DM$ can be re-expressed in terms of f_1, f_2 and f_3. These in turn can be written in terms of r and R and it can seen that

$$g_1 \approx \frac{1}{4R^3 k'^2} . \tag{6.13}$$

g_2 can be calculated in a similar way and in fact

$$g_2 \approx \frac{1}{8R^3} \left(2RM - 2 + \frac{R}{r} \right) . \tag{6.14}$$

We need to write $d\alpha_1$ and $d\alpha_2$ in terms of dr and dR. Since

$$\begin{aligned} \alpha_1 &= -k'^2 D^2 , \\ \alpha_2 &= -D^2 , \end{aligned} \tag{6.15}$$

this can be done if dk' is calculated. To do this we consider

$$\begin{aligned} -2dr &= df_3 \\ &= d(-D \operatorname{ns}_k DM) \\ &= -D \frac{\partial \operatorname{ns}_k DM}{\partial k} dk - D \frac{\partial \operatorname{ns}_k DM}{\partial D} dD - \operatorname{ns}_k DM dD \end{aligned} \tag{6.16}$$

and use [6, eq.710.56]

$$\frac{\partial \,\mathrm{ns}_k u}{\partial k} = \frac{\mathrm{cs}_k u \,\mathrm{ds}_k u}{k k'^2}\left[E(u;k) - k'^2 u - k^2 \,\mathrm{sn}_k u \,\mathrm{cd}_k u\right]. \tag{6.17}$$

This gives

$$\begin{aligned} d\alpha_1 &\approx -\frac{1}{2g_1(r^2-R^2)}\left(2dr - \frac{r^2-R^2}{R}MdR - 2\frac{r}{R}dR\right) - 8k'^2 RdR \\ d\alpha_2 &\approx -8RdR \end{aligned} \tag{6.18}$$

and so, while g_1 is order $1/k'^2$, $d\alpha_1$ is order k'^2. The asymptotic metric (6.5) can now be recovered by substitution.

7. Discussion

We began this calculation with two main motivations. Firstly, an explicit expression for the $(2,1)$ metric will be useful in the context of Hanany-Witten theory [18]; a $(2,1)$-monopole corresponds to the Hanany-Witten brane configuration illustrated in figure 4. Secondly, it would also be interesting if the $(2,1)$ metric or the $(1,2,1)$ metric could be used to conjecture an approximate form for the metric of an SU(2) 3-monopole, when two of the monopoles are close together and the other is far away.

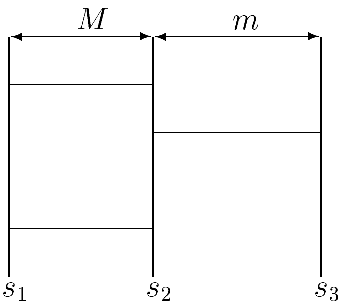
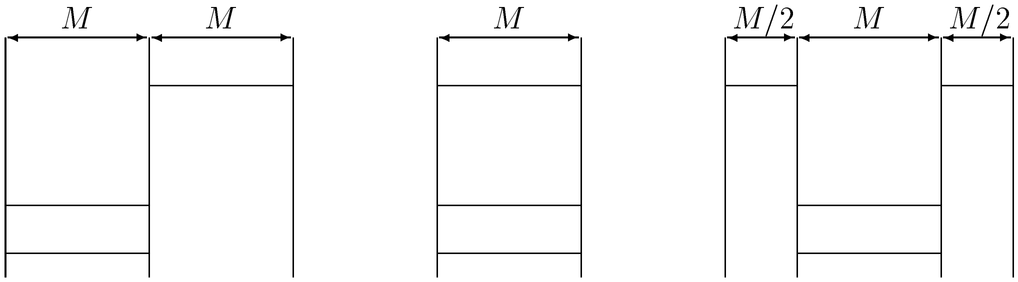


Figure 4: $(2,1)$ Hanany-Witten configuration with vertical fivebranes and horizontal threebranes.

If in a SU(2) 3-monopole, two of the monopoles are close together and one is far away, then the distant monopole has a well defined position and the two monopoles which are close together have a well defined centre of mass. This situation is not unlike that considered above. Figure 5 illustrates this likeness within the Hanany-Witten notation. In this notation, monopoles are replaced by threebranes which end on fivebranes. There are N fivebranes for an SU(N) monopole. The three dimensions of space are the codimensions in the fivebrane of the ends of the threebrane.

There are two types of interaction between the threebranes. There is a short range force acting between the whole length of vertically aligned threebranes and a long range force acting on the ends of the threebranes through the fivebranes. In figure 5(a) the (2,)-threebranes interact through both types of force, the (,1)-threebrane interacts with the (2,)-threebranes only through the fivebrane upon which they both end. In figure 5(b), there are two threebranes close together and one far away. To an exponential approximation, the far away threebrane only interacts with the other threebranes through the two fivebranes upon which they all end.



(a) (b) (c)

Figure 5: Hanany-Witten configurations illustrating the discussion of two monopoles close together and one far away. (a) corresponds to a $(2,1)$-monopole where $m = M$. (b) corresponds to a 3-monopole, the topmost threebrane is far away from the other two. (c) corresponds to a $(1,2,1)$-monopole.

Because of this, it might be expected that the moduli space of 3-monopoles with two monopoles close together and one far away, is exponentially well approximated by the $(2,1)$ metric.

In the 3-monopole, the threebranes end on the fivebranes from the same side and they all end on two different fivebranes. In the $(2,1)$-monopole, the $(2,\ )$-threebranes end on one side of a fivebrane and the $(\ ,1)$-threebrane on the other. This implies that for the approximation to work a sign and a factor of two must be changed in the $(2,1)$ metric. In the point dyon metric calculation of section 6.2, the correct change of sign results if DM is greater than $2K$, rather than less than it. The factor of two would be corrected if, instead of the $(2,1)$ metric, the $(1,2,1)$ metric was used with the $(1,\ ,\ )$-monopole constrained to be coincident with the $(\ ,\ ,1)$-monopole as illustrated in figure 5(c). The construction of the $(1,2,1)$ metric is discussed below. This picture is very vague, a more precise understanding of the relationship between asymptotic SU(2) metrics and SU(N) metrics would require the more sophisticated methods found in [4, 5].

We were also motivated to calculate the $(2,1)$ metric by [7]. Among the calculations in this paper, is an attempt to calculate the $(2,1)$ metric using the Legendre transform construction [20, 27]. The constraint equations arising in the Legendre transform prove intractable but the formulation itself is of great interest. Even without solving the constraint equation, Chalmers is able to study some features of the $(2,1)$ metric, for example, he is able to extract the point dyon metric. We hope that our work will prove useful in investigating the Legendre transform construction of the $(2,1)$ metric.

An interesting aspect of our calculation is the compelling form of the relative phase coordinate (2.39). It is the gauge invariant combination of the phase coordinates for the $(2,\ )$-monopole and the $(\ ,1)$-monopole. In section 4.1, this phase coordinate arises naturally when we use the hyperkähler quotient construction to construct the $(2,1)$ metric by attaching a 1-monopole to a $(2,[1])$-monopole. This use of the hyperkähler construction provides a simple method for constructing monopoles with no more than two monopoles of any type. In order to complete the tool box for such constructions, we have calculated the $([1], 2, [1])$ metric in section 8. From this, the $(1, 2, 1)$ metric could be calculated by choosing two suitable U(1) actions. As another example, three U(1) actions could be used to construct the $(2, 1, 2, 1)$ metric, by attaching a 1-monopole between the $(2, [1])$ metric and the $([1], 2, 1)$ metric. These methods could also be used to calculated moduli space of monopoles with other gauge groups like SO(5). Calculating the $(2, 2)$ metric by attaching the $(2, [1])$ metric to the $([1], 2)$ metric should not be too difficult either. It would simply require a larger action with which to perform the hyperkähler quotient.

The analysis of section 5.2 suggests that it is unlikely that there are bound geodesics on $\mathcal{M}^0_{(2,1)}$. An investigation of the quantum mechanics on $\mathcal{M}^0_{(2,1)}$ would be interesting but very difficult. The classical behaviour suggests the absence of bound states. In addition, S-duality predicts the absence of a Sen form on this space, because the form would be dual to W-bosons that are not seen at low energies [14, 33].

8. A space of SU(4) monopoles

In this section, the metric on a space of $([1], 2, [1])$-monopoles is calculated. The boundary condition breaks the symmetry from SU(4) to SU(2)×U(1)×SU(2). The asymptotic Higgs field lies in the gauge orbit of

$$\Phi_\infty = \begin{pmatrix} s_1 & & & \\ & s_1 & & \\ & & s_2 & \\ & & & s_2 \end{pmatrix}. \tag{8.1}$$

where $s_2 = -s_1$ and for convenience we choose $s_1 = -2$. Roughly speaking, a $([1], 2, [1])$-monopole is composed of two massive monopoles and two massless monopoles. The two massive monopoles are of the same type and are of different type to the two massless monopoles. These massless monopoles are of different types to each other. The relative moduli space $\mathcal{M}_{([1],2,[1])}$ is twelve dimensional, these twelve correspond to the separation of the massive monopoles, their SO(3) orientation in space, two sets of three SU(2) parameters corresponding to the unbroken gauge group and finally, two cloud parameters. In a $(1, 2, 1)$-monopole the $(1, \ , \ )$-monopole does not interact with the $(\ , \ , 1)$-monopole except in so far as each of them affects the $(\ , 2, \ )$-monopole. Since the two clouds in the $([1], 2, [1])$ case correspond to a massless $(1, \ , \ )$-monopole and a massless $(\ , \ , 1)$-monopole they should interact in a relatively simple manner.

In the limit, when one of the clouds is at infinity, $\mathcal{M}_{([1],2,[1])}$ reduces to $\mathcal{M}_{(2,[1])}$ with an additional infinite term. If both clouds are at infinity, it reduces to Atiyah-Hitchin but with two infinite terms. A restriction which amounts to identifying the two clouds, reduces the metric to the one considered in [30].

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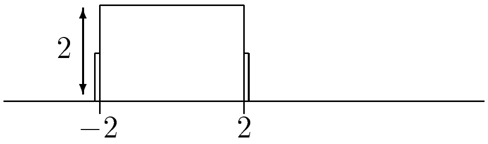
The general features of this SU(4) metric were discussed in [22]. The family of hyperkähler four-manifolds which are the hyperkähler quotients of $\mathcal{M}_{([1],2,[1])}$ were discussed in [10, 22]. Along with the previous calculation, the ease with which the monopole metric is found, once the Nahm equations are solved, shows that the monopole metric problem for larger groups is no more difficult than the problem when the group is SU(2).

The $([1],2,[1])$ Nahm data are 2×2 skew-hermitean matrix functions of $s\in[-2,2]$. They satisfy the Nahm equation and are analytic over the entire closed region. They are represented by the diagram



(8.2)

The gauge action is $G^{0,0}$. There are two SU(2) actions, one given by $G^{0,\star}/G^{0,0}$ and the other by $G^{\star,0}/G^{0,0}$. There is also a translational action, like the one discussed above and a rotational action given by

$$\begin{aligned} T_0 &\to T_0, \\ T_i &\to \sum_j R_{ij}T_j\,. \end{aligned} \tag{8.3}$$

The centre of mass and the U(1) action given by $\exp(if(s)\mathbf{1}_2)$ are fixed in the same way as before. This means that the Nahm data are traceless.

As before, the Nahm equations are reduced by the ansatz

$$\begin{aligned} T_0(s) &= 0, \\ T_i(s) &= f_i(s)e_i \end{aligned} \tag{8.4}$$

to the Euler-Poinsot top equations. The solutions are Euler top functions but with weaker boundary conditions:

$$\begin{aligned} f_1(s) &= \pm\frac{D\mathrm{cn}_k D(s+\tau)}{\mathrm{sn}_k D(s+\tau)}\,, \\ f_2(s) &= \pm\frac{D\mathrm{dn}_k D(s+\tau)}{\mathrm{sn}_k D(s+\tau)}\,, \\ f_3(s) &= \pm\frac{D}{\mathrm{sn}_k D(s+\tau)}\,, \end{aligned} \tag{8.5}$$

with $0\leq k\leq 1$, $\tau>2$ and $D(\tau+2)<2K(k)$. The extra parameter τ reflects the analyticity of the Nahm data at $s=-2$. All signs are negative or exactly two signs are positive. More solutions are given by sending s to $-s$ and changing the signs of f_i.

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Using these solutions to the Nahm equations, it is a simple exercise to find the metric on $\mathcal{M}_{([1],2,[1])}$. The metric depends on D, k and τ and three sets of SU(2) coordinates. Each of the SU(2)'s acts isometrically, so it is only necessary to calculate the metric in the neighbourhood of the identity of the SU(2)'s. The SU(2) actions are then used to find the metric at a general point. The method is a duplicate of that used in [9, 26] and the interested reader is referred to these papers.

Coordinates on the quotient space $\mathcal{M}_{([1],2,[1])}/(\mathrm{SO}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(2))$ are provided by

$$\begin{aligned} \alpha_1 &= k'^2 D^2\,, \\ \alpha_2 &= D^2\,, \\ \alpha_6 &= \sum_i [f_i^2(2) - f_i^2(-2)]\,. \end{aligned} \tag{8.6}$$

The remaining nine coordinates on $\mathcal{M}_{([1],2,[1])}$ are given by SU(2) matrices corresponding to rotations $R \in$ SO(3) and group actions given by $G(2) \in$ SU(2) for $G^{0,\star}/G^{0,0}$ and $G(-2) \in$ SU(2) for $G^{\star,0}/G^{0,0}$. The metric is written in terms of left-invariant one-forms given by

$$\begin{aligned} \frac{i}{2}\tau_i\sigma_i &= R^\dagger dR\,, \\ \frac{i}{2}\tau_i\check{\sigma}_i &= G(2)^\dagger dG(2)\,, \\ \frac{i}{2}\tau_i\hat{\sigma}_i &= G(-2)^\dagger dG(-2)\,. \end{aligned} \tag{8.7}$$

where it should be noted that $\check{\sigma}$ and $\hat{\sigma}$ are different symbols. Because the SU(2) actions are isometric, the coefficients of the metric depend only on D, k and τ. In order to express the metric, it is useful to define the following functions of D, k and τ;

$$\begin{aligned} g_1(k,D,\tau) &= \int_{-2}^{2} \frac{1}{f_2^2}\,ds\,, & g_2(k,D,\tau) &= \int_{-2}^{2} \frac{1}{f_3^2}\,ds\,, \\ A(k,D,\tau) &= f_1(2)f_2(2)f_3(2)\,, & B(k,D,\tau) &= f_1(-2)f_2(-2)f_3(-2)\,, \\ X(k,D,\tau) &= AB(g_1+g_2) + B - A\,. \end{aligned} \tag{8.8}$$

The following combination of one-forms also simplifies the expression for the metric

$$\begin{aligned} \omega_1 &= \check{\sigma}_1 - \frac{f_3(2)}{f_2(2)}\sigma_1\,, \\ \omega_2 &= \check{\sigma}_2 - \frac{f_1(2)}{f_3(2)}\sigma_2\,, \\ \omega_3 &= \check{\sigma}_3 - \frac{f_1(2)}{f_2(2)}\sigma_3 \end{aligned} \tag{8.9}$$

and

$$\rho_1 = \hat{\sigma}_1 - \frac{f_3(-2)}{f_2(-2)}\sigma_1 \,,$$
$$\rho_2 = \hat{\sigma}_2 - \frac{f_1(-2)}{f_3(-2)}\sigma_2 \,,$$
$$\rho_3 = \hat{\sigma}_3 - \frac{f_1(-2)}{f_2(-2)}\sigma_3 \,. \tag{8.10}$$

The metric is then given by the following complicated expression:

$$\begin{aligned} ds^2 = & \frac{AB}{4(B-A)}(g_1 d\alpha_1 + g_2 d\alpha_2)^2 + \frac{1}{4}(g_1 d\alpha_1^2 + g_2 d\alpha_2^2) + \frac{1}{A-B} d\alpha_6^2 + \\ & + \frac{g_1(ABg_2 + B - A)k^4 D^4}{X}\sigma_1^2 + \frac{(g_1+g_2)g_2 D^4}{g_1}\sigma_2^2 + \frac{(g_1+g_2)g_1 k'^4 D^4}{g_2}\sigma_3^2 + \\ & + \frac{A^2(B(g_1+g_2)-1)}{f_1(2)^2 X}\omega_1^2 + \frac{Ag_1+1}{f_2(2)^2 g_1}\omega_2^2 + \frac{Ag_2+1}{f_3(2)^2 g_2}\omega_3^2 - \\ & - \frac{B^2(A(g_1+g_2)+1)}{f_1(-2)^2 X}\rho_1^2 - \frac{Bg_1-1}{f_2(-2)^2 g_1}\rho_2^2 - \frac{Bg_2-1}{f_3(-2)^2 g_2}\rho_3^2 + \\ & + \frac{2ABg_1 k^2 D^2}{f_1(2)X}\sigma_1\omega_1 + \frac{2D^2 g_2}{g_1 f_2(2)}\sigma_2\omega_2 + \frac{2D^2 g_1 k'^2}{g_2 f_3(2)}\sigma_3\omega_3 - \\ & - \frac{2ABg_1 k^2 D^2}{f_1(-2)X}\sigma_1\rho_1 - \frac{2D^2 g_2}{g_1 f_2(-2)}\sigma_2\rho_2 - \frac{2D^2 g_1 k'^2}{g_2 f_3(-2)}\sigma_3\rho_3 + \\ & + \frac{2AB}{f_1(2)f_1(-2)X}\omega_1\rho_1 - \frac{2}{g_1 f_2(2) f_2(-2)}\omega_2\rho_2 - \frac{2}{g_2 f_3(2) f_3(-2)}\omega_3\rho_3 . \end{aligned} \tag{8.11}$$

The Sp(4) condition [30] can be imposed: $f_1(2) = -f_1(-2)$, $f_2(2) = f_2(-2)$ and $f_3(2) = f_3(-2)$, along with the identifications $\check{\sigma}_1 = \hat{\sigma}_1$, $\check{\sigma}_2 = -\hat{\sigma}_2$, $\check{\sigma}_3 = -\hat{\sigma}_3$. This reduces (8.11) to the metric found in [30]. Alternatively, the limit $A \to \infty$ or $B \to \infty$ can be taken and it can be shown that this reduces the $([1], 2, [1])$ metric to the $(2, [1])$ metric discussed in [9, 26], along with an infinite term corresponding to the moment of inertia of the cloud at infinity. If $A \to \infty$ and $B \to \infty$, the metric reduces to the Atiyah-Hitchin metric along with infinite terms corresponding to the inertia of both clouds at infinity.

There is a three-dimensional geodesic submanifold of $\mathcal{M}^{([1],2,[1])}$ obtained by imposing D_2 symmetry on the monopoles. This was denoted $\boldsymbol{X}$ in [22] and is described there. It is the $([1], 2, [1])$ analogue of the Y space of Dancer and Leese [12]. Imposing spherical symmetry on $\boldsymbol{X}$ reduces to one-dimensional submanifolds whose Nahm data are

$$f_1(s) = f_2(s) = f_3(s) = -\frac{1}{s+\tau} \,, \tag{8.12}$$

where $\tau > 2$ or $\tau < -2$. Using the expression for the metric, we can determine the geodesics in this one-dimensional example. If $\tau > 2$ and initially decreasing, then it continues to approach τ=2 without ever reaching there. This corresponds to one of the clouds increasing to arbitrarily large radius with the other cloud remains small. The massive monopoles are at the origin. If $\tau > 2$ and initially increasing, then it reaches $\tau = \infty$ in finite time and then, since $\tau = \infty$ and $\tau = -\infty$ are equivalent by (8.12), τ increases from $-\infty$ to approach -2. If τ is initially close to two, this represents a process where one cloud is initially large and decreasing. It reaches its minimum size and then the other cloud continually increases from its minimum size to arbitrarily large radius.

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